Math 235 Midterm 2 Solution Spring 2001

1. (24 points) You are given below the matrix A together with its row reduced echelon form B

$$A = \begin{pmatrix} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 2 & 4 & 2 & 2 & 2 \\ 2 & 1 & 4 & -1 & 1 & 0 \\ 1 & 1 & 3 & 0 & 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Note: you do **not** have to check that A and B are indeed row equivalent.

a) (8 points) Find a basis for the null space $Null(A)$ of A.

Answer: We use the row reduced echelon form to write, in parametric form, the general solution of $A\vec{x} = \vec{0}$ (i.e., the general vector in $Null(A)$). The free variables are: x_3, x_4, and x_5. Hence, the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

and the three column vectors above are a basis $Null(A)$.

b) (8 points) Find a basis for the column space of A.

Answer: The pivot columns of A are a basis for col(A). By definition, the pivot positions of A are the same as those of its reduced echelon form B. Hence, the first, second and six-th columns of A are a basis for col(A):

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

c) (8 points) Is the vector $w := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ in the column space of A? Use part b to

Justify your answer!

Answer: The vector w is in the column space of A, if and only if it is a linear combination of the basis vectors we found in part b. This is the case, if and only if the augmented matrix below represents a *consistent* system. Row reduction yields:

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The system is inconsistent, since we have a pivot in the rightmost column. We conclude that w does **not** belong to col(A).

2. (16 points) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$.

   Compute i) A^{-1}, ii) B^{-1}, and iii) $(AB)^{-1}$. Check your answers in parts i and ii by calculating AA^{-1} and BB^{-1}.

   Answer: i) (6 points) We find A^{-1} by row reducing the 3×6 matrix obtained by augmenting A by the identity matrix. Interchanging the second and third rows, we get

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

   We see that A^{-1}, whose columns are the last three columns of the 3×6 reduced echelon form, is equal to A.

   ii) (6 points) $[B \mid I] = \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & -2 & -5 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$.

   So, $B^{-1} = \begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

   iii) (4 points) $(AB)^{-1} = B^{-1}A^{-1} = \begin{pmatrix} 1 & -2 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -5 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

3. (16 points) a) (8 points) Let A, B and C be 4×4 matrices satisfying

$$B = ACA^{-1} + 2A,$$

   with A invertible. Solve for C in terms of A and B.

   Answer:

$$\begin{aligned} B &= ACA^{-1} + 2A && \text{Subtract } 2A \text{ from both sides} \\ B - 2A &= ACA^{-1} && \text{Multiply both sides by } A \text{ on the right} \\ (B - 2A)A &= AC && \text{Multiply both sides by } A^{-1} \text{ on the left} \\ A^{-1}(B - 2A)A &= C \end{aligned}$$

   Simplifying, we get $C = A^{-1}BA - 2A$.

   b) (7 points) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$. Find a matrix B satisfying $AB = C$.

   Answer: $B = A^{-1}C = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 8 & 17 \\ -2 & -3 & -7 \end{bmatrix}$

   c) (1 point) Check your answer in part b by calculating AB.

4. (18 points) Determine which of the following sets in $\mathbb{R}^n$ is a subspace. If it is not, **find** a property in the definition of a subspace which this set violates. If it is a subspace, **find** a matrix A such that this set is either $Null(A)$ or $Col(A)$.

(a) $$\left\{ \begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - 2x_3 \\ 2x_3 - x_1 \\ 5x_1 + 3x_2 - x_3 \end{bmatrix} \ : \ x_1, x_2, x_3 \text{ are arbitrary real numbers} \right\}$$

Answer

We can write the general element as a linear combination
$$\begin{bmatrix} x_1 + 3x_3 \\ 3x_2 - 2x_3 \\ 2x_3 - x_1 \\ 5x_1 + 3x_2 - x_3 \end{bmatrix} = x_1 \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 5 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \end{bmatrix} + x_3 \cdot \begin{bmatrix} 3 \\ -2 \\ 2 \\ -1 \end{bmatrix}.$$ Hence, this subset is a subspace, which is equal to the column space of the matrix $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 3 & 2 \\ -1 & 0 & 2 \\ 5 & 3 & -1 \end{bmatrix}$.

(b) $$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \ : \ x_1, x_2, x_3, x_4 \ \text{ are real numbers satisfying } \begin{array}{rcl} x_1 + x_2 + x_3 & = & x_4 \\ 5x_2 & = & 4x_3 \end{array} \right\}$$

Answer: This subset of $\mathbb{R}^4$ is the general solution of the system of homogeneous linear equations $\begin{array}{rcl} x_1 + x_2 + x_3 - x_4 & = & 0 \\ 5x_2 - 4x_3 & = & 0 \end{array}$. It is hence a subspace which is equal to the Null space of $\begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 5 & -4 & 0 \end{bmatrix}$.

(c) The unique plane in $\mathbb{R}^3$ passing through the three points
$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Answer: There is a unique plane P passing through $\{v_1, v_2, v_3\}$ (it is not their span!). The plane P does not pass through the origin. Hence it is **not** a subspace. The plane P does not contain the zero vector, because the three vectors given are linearly independent (check by row reduction).

5. (16 points) a) (6 points) Compute the area of the **triangle** in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,2)$, $(3,1)$. *Hint*: Find a parallelogram, whose area is twice that of the triangle.

Answer: The area of this triangle is half the area of the parallelogram with vertices $(0,0)$, $(1,2)$, $(3,1)$, and $(4,3)$. The area of the parallelogram is the absolute value of the determinant
$$\det \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = 1 - 6 = -5.$$

Hence, the are of the triangle is $\frac{5}{2}$.

b) (6 points) Compute the volume of the parallelepiped in $\mathbb{R}^3$ with vertices $\vec{0}$, v_1, v_2, v_3, $v_1 + v_2$, $v_1 + v_3$, $v_2 + v_3$, $v_1 + v_2 + v_3$, where

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

Answer: The volume is the absolute value of the determinant of the 3×3 matrix M with columns v_1, v_2 and v_3.

$$\begin{vmatrix} 2 & 2 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -5 \end{vmatrix} = -10$$

Hence, the volume is 10.

c) (4 points) Use your answer in part (b) and the algebraic properties of determinants to compute the volume of the parallelepiped obtained if v_1, v_2, v_3 are replaced by w_1, w_2, w_3, where $w_i = 2v_i$, for $i = 1, 2, 3$.

Answer: The determinant of the matrix gets multiplied by 2 each time we multiply a column (or row) by 2. Hence, the determinant of the matrix $[w_1 w_2 w_3]$ is 2^3 times that of $[v_1 v_2 v_3]$.

$$\det[w_1 w_2 w_3] \;=\; 8 \times \det[v_1 v_2 v_3] \;=\; 8(-10) \;=\; -80.$$

Hence, the volume is 80.

6. (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2. Recall that a vector in $\mathbb{P}_2$ is a polynomial $p(t)$ of the form $p(t) = a_0 + a_1 t + a_2 t^2$, where the coefficients a_0, a_1, a_2 are arbitrary real numbers.

   (a) Find a polynomial $p(t)$, of degree at most 2, satisfying $p(0) = 4$, $p(1) = 1$, and $p(2) = 0$.

   Answer: The three equations $p(0) = 4$, $p(1) = 1$, and $p(2) = 0$, translate to equations in terms of the coefficients of $p(t)$:

$$\begin{aligned} a_0 + a_1 \cdot 0 + a_2 \cdot 0^2 &= 4 \\ a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 &= 1 \\ a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 &= 0 \end{aligned}$$

   This is a system of linear equations in the variables a_0, a_1, a_2. Row reduction yields

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

   Hence, $a_0 = 4$, $a_1 = -4$, $a_2 = 1$, and the polynomial $p(t)$ is equal to $4 - 4t + t^2$.

(b) The subset H of $\mathbb{P}_2$, of polynomials $p(t)$ of degree ≤ 2, which in addition satisfy

$$p(2) = 0$$

is a *subspace* of $\mathbb{P}_2$. (You may assume this fact). Find a basis for H. **Explain** why the set you found is linearly independent and why it spans H.

Answer: H is the subset of polynomials, which satisfy the equation

$$a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 \quad = \quad 0.$$

Regarding it as a system of one equation in three variables, we get that a_1 and a_2 are free and

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -2a_1 - 4a_2 \\ a_1 \\ a_2 \end{bmatrix} = a_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}.$$

If we choose $a_1 = 1$ and $a_2 = 0$, we get the polynomial $-2 + t$. If we choose $a_1 = 0$ and $a_2 = 1$, we get the polynomial $-4 + t^2$. Hence, $\{-2 + t, -4 + t^2\}$ is a basis for H.