Solution of Midterm 2, Fall 2015

1. (20 points) You are given below the matrix A together with its row reduced echelon form B (you need *not* verify that B is indeed the reduced echelon form of A)

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & -2 & 1 \end{pmatrix} \qquad B = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

x_3, x_5, x_6 free

a) Find a basis for the null space $Null(A)$ of A. Justify!

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -x_3 - x_5 + x_6 \\ -2x_3 + x_6 \\ x_3 \\ x_5 - x_6 \\ x_5 \\ x_6 \end{pmatrix} = x_3 \underbrace{\begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + x_5 \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{v_2} + x_6 \underbrace{\begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}_{v_3}$$

The set $\{v_1, v_2, v_3\}$ is a basis for $Null(A)$

b) Find a basis for the column space $Col(A)$ of A. Justify!

The pivot columns (first, second, and fourth) form a basis for $Col(A)$

Basis: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}$

c) Is the sixth column $a_6 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ of the matrix A in part 1a) a linear combination of the first five columns of A? Justify your answer. Hint: A careful reading of Question 1 will eliminate the need for any computations.

Let b_j be the j-th column of B.

Then $b_6 = -b_1 - b_2 + b_4$.

Hence $a_6 = -a_1 - a_2 + a_4$.

In particular, a_6 <u>is</u> a linear combination of a_1, a_2, a_3, a_4, a_5.

2. (a) (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x_1, x_2) = (3x_1 - 4x_2, -5x_1 + 7x_2).$$

Show that T is invertible and find a formula for T^{-1}.

$$T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 3x_1 - 4x_2 \\ -5x_1 + 7x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & -4 \\ -5 & 7 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$A^{-1} = \begin{pmatrix} 7 & 4 \\ 5 & 3 \end{pmatrix}$ and so

$$T^{-1}\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = A^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7x_1 + 4x_2 \\ 5x_1 + 3x_2 \end{pmatrix}$$

$$T^{-1}(x_1, x_2) = (7x_1 + 4x_2, 5x_1 + 3x_2)$$

(b) (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2. Recall that a vector in $\mathbb{P}_2$ is a polynomial $p(x)$ of the form $p(x) = a_0 + a_1 x + a_2 x^2$, where the coefficients a_0, a_1, a_2 are arbitrary real numbers. Is the subset $\{f, g, h\}$ of $\mathbb{P}_2$, consisting of the three polynomials $f(x) = 1 - x$, $g(x) = 1 - x^2$, and $h(x) = 1 + x + x^2$, linearly dependent or independent? Justify your answer.

Solve for c_1, c_2, c_3 in the vector equation (in $\mathbb{P}_2$)

$$\underbrace{c_1(1-X) + c_2(1-X^2) + c_3(1+X+X^2)}_{(c_1+c_2+c_3)+(-c_1+c_3)X+(-c_2+c_3)X^2} = 0 + 0X + 0X^2 \quad (*)$$

Equating coefficients we get the system of 3 linear eq

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 + c_3 &= 0 \\ -c_2 + c_3 &= 0 \end{aligned}$$

whose matrix form is

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

has a pivot in every column. Hence the matrix equation has a unique solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Hence, equation (*) has only the trivial solution and the three polynomials are linearly independent (by definition of linear independence).

3. a) (8 points) Let A, B, and C be invertible $n \times n$ matrices. Show that there exists precisely one $n \times n$ matrix X satisfying $C(A+X)B = A$. Express X in terms of A, B, and C.

$$A + X = C^{-1} A B^{-1}$$

$$X = C^{-1} A B^{-1} - A$$

b) (12 points) Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Compute its inverse A^{-1}. (**Check** that $AA^{-1} = I$.)

Add $-2R_1$ to R_2

Interchange R_2 and R_3

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right) \sim$$

Multiply new R_2 by -1

$$\sim \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & +1 & 0 & 0 & 0 & -1 \\ 0 & 2 & 1 & -2 & 1 & 0 \end{array}\right)$$

Add $-2R_2$ to R_3

$\sim$

Add R_2 to R_1

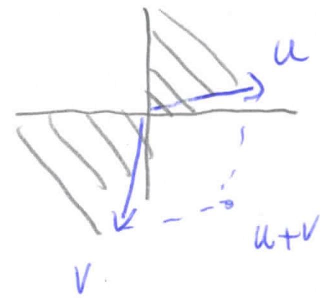
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}$$

4. (20 points) Determine if the following subset H of $\mathbb{R}^n$ is a subspace. If it is not, find a property in the definition of a subspace which H violates. If H is a subspace find either a set of vectors which spans it, or a matrix A such that H is $Null(A)$ (which will provide the justification that it is indeed a subspace).

(a) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ such that } xy \geq 0 \right\}$ (the union of the first and third quadrants).

H is not a subspace because it is not closed under addition.



$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ are in H,

but $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is not in H.

(b) $H = \left\{ \begin{bmatrix} 4x_1 + x_3 \\ 2x_1 - 3x_2 \\ x_2 + 6x_3 \end{bmatrix} \text{ such that } x_1, x_2, x_3 \text{ are arbitrary real numbers} \right\}$

$$= x_1 \underbrace{\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}}_{v_1} + x_2 \underbrace{\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}}_{v_2} + x_3 \underbrace{\begin{pmatrix} 1 \\ 0 \\ 6 \end{pmatrix}}_{v_3}$$

$H = \text{Span}\{v_1, v_2, v_3\}$.

Hence, H is a subspace.

5. (20 points) a) Compute the volume of the parallelepiped in $\mathbb{R}^3$ with vertices $\vec{0}$, v_1, v_2, v_3, $v_1 + v_2$, $v_1 + v_3$, $v_2 + v_3$, $v_1 + v_2 + v_3$ (the parallelepiped determined by v_1, v_2, and v_3) where

$$v_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\text{Vol} = \left| \det \begin{pmatrix} -1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \right| \underset{\uparrow}{=} \left| \det \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 4 \\ 0 & 2 & 4 \end{pmatrix} \right| = \left| -1 \underbrace{(3 \cdot 4 - 2 \cdot 4)}_{4} \right|$$

Add $2R_1$ to R_2
Add R_1 to R_3

$= 4.$

b) Let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ sending a vector $\vec{x}$ to $A\vec{x}$, where A is the matrix $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. Suppose v_1, v_2 are two vectors in $\mathbb{R}^2$, such that the parallelogram with vertices 0, v_1, v_2, v_1+v_2 has area 8 square meters. Compute the area of the image of this parallelogram under the transformation T. (The image is the parallelogram with vertices 0, $A(v_1)$, $A(v_2)$, and $A(v_1+v_2)$). **Justify your answer!**

$$\text{area} = |\det(Av_1 \;\; Av_2)| = |\det(\underset{\substack{\uparrow \\ \text{matrix} \\ \text{product}}}{A\cdot} \underbrace{(v_1 \; v_2)}_{\substack{\text{the } 2\times 2 \\ \text{matrix with} \\ \text{columns} \\ v_1 \text{ and } v_2}})| =$$

$$= \underbrace{|\det(A)|}_{(10-3)} \underbrace{|\det(v_1 v_2)|}_{= 8} = 7 \cdot 8 = 56$$