Math 235 Midterm 1 Solution Spring 2005

1. (20 points) a) Find the row **reduced** echelon augmented matrix of the system
$x_1 + x_2 + x_3 + x_4 = 4$
$x_2 - x_3 + 2x_4 + x_5 = 3$
$x_1 + 2x_2 + 5x_4 + x_5 = 9$

Answer: The row reduction takes 5 elementary operations:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 2 & 1 & 3 \\ 1 & 2 & 0 & 5 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 2 & 1 & 3 \\ 0 & 1 & -1 & 4 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 & 0 & 2 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 4 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \sim$$
$$\begin{bmatrix} 1 & 0 & 2 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

b) Find the general solution for the system.

Answer: x_3 and x_5 are free variables.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 + x_5 + 2 \\ x_3 - x_5 + 1 \\ x_3 \\ 1 \\ x_5 \end{bmatrix} = x_3 \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

2. (18 points) Let $u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 3 \\ 2 \\ h \end{bmatrix}$, and $u_4 = \begin{bmatrix} h \\ 1 \\ 1 \end{bmatrix}$.

Justify your answers to the following questions!

a) For which real numbers h does the set $\{u_1, u_2, u_3, u_4\}$ span the whole of $\mathbb{R}^3$?

Answer: Let A be the matrix, whose columns are the vectors u_i. The question is equivalent to:

"For which value of h does the matrix A have a pivot in every row?"

Row reduction yields that A is row equivalent to the following matrix in echelon form:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & -1 & h-2 \\ 0 & 0 & h-3 & 1-h \end{bmatrix} \tag{1}$$

If $h \neq 3$, then we get a pivot in the $(3,3)$ position (third row and third column). If $h = 3$, then we get a pivot in the $(3,4)$ position. Thus, for every value of h, we get a pivot in every row. Consequently, the set $\{u_1, u_2, u_3, u_4\}$ span the whole of $\mathbb{R}^3$, *for every value of h.*

b) For which values of h does the vector u_3 belong to the plane spanned by $\{u_1, u_2\}$

Answer: Precisely when $h = 3$, for the following reason. The question is equivalent to:

"For which value of h is the vector equation $x_1\vec{u}_1 + x_2\vec{u}_2 = \vec{u}_3$ consistent?"

The coefficient matrix of this equation has columns u_1 and u_2, and the augmented matrix of this equation is the 3×3 matrix B, whose columns are u_1, u_2, and u_3. The row echelon matrix of B is obtained by considering the first three columns of the matrix (1) above. We get a pivot in the rightmost column (and the system is inconsistent) if and only if $h \neq 3$. Thus, the system is *consistent* if and only if $h = 3$.

c) For which values of h does the vector u_4 belong to the plane spanned by $\{u_1, u_2\}$?

Answer: Precisely when $h = 1$, for a reason similar to part b (consider the first, second, and fourth columns of the matrix (1) above).

d) For which real numbers h is the set $\{u_1, u_2, u_3, u_4\}$ linearly independent?

Answer: This set is always linearly *dependent*! Four vectors in $\mathbb{R}^3$ are always linearly dependent (there are more vectors than entries in each vector.)

3. (13 points) Set up a system of linear equations for finding the electrical **branch** currents $I_1, ..., I_6$ in the following circuit using i) the junction rule: the sum of currents entering a junction is equal to the sum of currents leaving the junction. ii) Ohm's rule: The drop in the voltage ΔV across a resistance R is related to the (directed) current I by the equation $\Delta V = IR$. iii) Kirchhof's circuit rule: the sum of the voltage drops due to resistances around any closed loop in the circuit equals the sum of the voltages induced by sources along the loop. *Note: Do* **not** *solve the system.*

4. (16 points) Determine if the statement is true or false. If it is true, give a reason. If it is false, provide a counter example. (credit will be given only if a valid justification is provided).

   (a) If A is a 4×3 matrix (4 rows and 3 columns), $\vec{b}$ is a vector in $\mathbb{R}^4$, and the equation $A\vec{x} = \vec{b}$ is consistent, then it has infinitely many solutions.

   Answer: False. As a counter example consider the following equation, which has a unique solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

   (b) Let A be a square 3×3 matrix. If the equation $A\vec{x} = \vec{b}$ is consistent, for all vectors $\vec{b}$ in $\mathbb{R}^3$, then the columns of A are linearly independent.

   Answer: True. Reason:

The equation $A\vec{x} = \vec{b}$ is consistent, for all vectors $\vec{b}$ in $\mathbb{R}^3 \Longrightarrow$
The augmented matrix $[A \mid \vec{b}]$ does not have a pivot in the rightmost column for all vectors $\vec{b}$ in $\mathbb{R}^3 \Longrightarrow$
A has a pivot in every row $\Longrightarrow$
A has a pivot in every column (A has the same number of rows and columns) $\Longrightarrow$ The columns of A are linearly independent

(c) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$. For every three vectors v_1, v_2, v_3, in $\mathbb{R}^2$, the set $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent (in $\mathbb{R}^3$).

Answer: True. Reason:

The set of three vectors $\{v_1, v_2, v_3\}$ in $\mathbb{R}^2$ is linearly dependent (there are more vectors than entries in each vector). Hence, the vector equation
$x_1v_1 + x_2v_2 + x_3v_3 = \vec{0}$
has a non trivial solution. Evaluating T on both sides, using the properties of linear transformations, we get that the same non-trivial solution solves also the equation

$$x_1T(v_1) + x_2T(v_2) + x_3T(v_3) \quad = \quad \vec{0}$$

(of vectors in $\mathbb{R}^3$). Hence, the set $\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent.

(d) Let A be a 3×4 matrix and b_1, b_2 two vectors in $\mathbb{R}^3$. If the vector equations $A\vec{x} = b_1$ and $A\vec{x} = b_2$ are both consistent, then so is the equation $A\vec{x} = b_1 - b_2$.

Answer: True. Reason:

If u is a solution of $Ax = b_1$ and v is a solution of $Ax = b_2$ then $u - v$ is a solution of $Ax = b_1 - b_2$, by the properties of matrix multiplication:

$A(u - v) = Au - Av = b_1 - b_2$.

5. (15 points) a) Find two vectors v_1, v_2 in $\mathbb{R}^3$ which span the plane given by the equation

$$x_1 + 3x_2 - x_3 \;=\; 0.$$

Answer: The plane is the set of solutions of this single linear equation. The variables x_2 and x_3 are free and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

This expression shows that the general solution is precisely the plane spanned by the two column vectors on the right hand side.

b) Let v_1, v_2 be the two vectors from part a). Find the equation of the plane consisting of all vectors of the form $sv_1 + tv_2 + \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, where s, t are real numbers.

Answer: The above is a parametrization of the plane $x_1 + 3x_2 - x_3 = b$, parallel to the one in part a), passing through the particular vector $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$. Plug $x_1 = 2$, $x_2 = 1$, $x_3 = -1$ to get that $b = 6$. So the equation of the plane is: $x_1+3x_2-x_3 = 6$.

6. (18 points) Find the standard matrix of each of the following linear transformations.

a) T is the map from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$T(x_1, x_2, x_3) = (2x_1 + x_2 - x_3, 5x_1 + 2x_2 + x_3, 9x_1 + 7x_2 - 5x_3)$.

Answer: $\begin{bmatrix} 2 & 1 & -1 \\ 5 & 2 & 1 \\ 9 & 7 & -5 \end{bmatrix}$.

b) T is the map from $\mathbb{R}^2$ to $\mathbb{R}^2$, which rotates points (about the origin) through $3\pi/4$ radians (counterclockwise).

We determine the standard matrix $A = [\vec{a}_1\vec{a}_2]$ of T column by column:

$$\begin{aligned} \vec{a}_1 &= T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(3\pi/4) \\ \sin(3\pi/4) \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \\ \vec{a}_2 &= T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin(3\pi/4) \\ \cos(3\pi/4) \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \end{aligned}$$

Thus, $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

c) T is the map from $\mathbb{R}^2$ to $\mathbb{R}^2$, which first reflects points through the vertical x_2 axis and then reflects points through the line $x_2 = x_1$.

Answer: Denote by R_1 the reflection through the vertical x_2 axis and by R_2 the reflection through the line $x_2 = x_1$. We determine the standard matrix $A = [\vec{a}_1\vec{a}_2]$ of T column by column:

$$\begin{aligned} \vec{a}_1 &= T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = R_2\left(R_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = R_2\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\ \vec{a}_2 &= T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = R_2\left(R_1\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = R_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}$$

Thus, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.