

Math 235 Midterm 1 Fall 2015
 SOLUTION

1. (20 points) a) Find the row reduced echelon augmented matrix of the system

$$x_1 + 2x_2 + 3x_3 + 3x_4 + 5x_5 = 10$$

$$x_1 + 3x_2 + 5x_3 + 4x_4 + 7x_5 = 14$$

$$2x_1 + 5x_2 + 8x_3 + 7x_4 + 13x_5 = 25$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 3 & 5 & 10 \\ 1 & 3 & 5 & 4 & 7 & 14 \\ 2 & 5 & 8 & 7 & 13 & 25 \end{array} \right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 3 & 5 & 10 \\ 0 & 1 & 2 & 1 & 2 & 4 \\ 0 & 1 & 2 & 1 & 3 & 5 \end{array} \right) \sim$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 3 & 0 & 5 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right) \sim \left(\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right)$$

- b) Find the general solution for the system.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_3 - x_4 + 1 \\ -2x_3 - x_4 + 2 \\ x_3 \\ x_4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

c) Find the general solution of the associated homogeneous system (replacing the constant, on the right hand side of the equations in part a, by zeros). Try to avoid computations. Justify your answer!

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

d) Find two vectors which span the solution set in Part c.

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Since every solution is a linear combination
of these two vectors,

2. (20 points) Let $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} -1 \\ 2 \\ h \end{bmatrix}$.

a) For which real numbers h is the vector $\vec{b} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ in $\text{span}\{v_1, v_2, v_3\}$? Justify your answer!

The vector $\vec{b} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ has augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & 0 & 2 & 2 \\ 3 & 1 & h & 0 \end{array} \right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -2 & +4 & 6 \\ 0 & 0 & h-1 & 0 \end{array} \right)$$

The system is consistent for all h , since we can never have a pivot in the rightmost column.

(indeed, $\vec{b} = v_1 - 3v_2$).

b) For which values of h do the three vectors $\{v_1, v_2, v_3\}$ span the whole of \mathbb{R}^3 ? Justify your answer!

$h \neq 1$, In that case we have a pivot in every row of the matrix $(v_1 \ v_2 \ v_3)$.

3. (20 points) For each of the following maps, determine if it is a linear transformation. If it is, find its standard matrix. If it is not, explain which property of linear transformations it violates (and why it violates it).

a) T is the map from \mathbb{R}^3 to \mathbb{R}^3 defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, x_1 + 2x_3, x_2 + 6x_3).$$

$$T(\vec{x}) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

The Standard matrix of T

b) T is the map from \mathbb{R}^3 to \mathbb{R}^2 defined by

$$T(x_1, x_2, x_3) = (x_1 + x_2, x_1 + 2x_3, 2x_1 - 3x_3).$$

Not linear

$$\text{Let } v = (1, 0, 0)$$

$$T(v) = (1, 0, 2)$$

$$T(2v) = (2, 0, 4) \neq 2T(v).$$

c) T is the map from \mathbb{R}^2 to \mathbb{R}^2 , which sends each vector to its reflection with respect to the line $x_2 = -x_1$ (please, pay attention to the signs!).

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Standard matrix

$$A = \begin{pmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} & T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

4. (20 points)

- (a) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 4 \\ 1 & k \end{pmatrix}$. What value of k , if any, will make $AB = BA$? Justify your answer!

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 1 & k \end{pmatrix} = \begin{pmatrix} 5 & * \\ * & * \end{pmatrix} \quad \text{The } (1,1) \text{ entry is } 5$$

H

$$BA = \begin{pmatrix} 3 & 4 \\ 1 & k \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 15 & * \\ * & * \end{pmatrix} \quad \text{The } (1,1) \text{ entry is } 15$$

$AB \neq BA$ for all values of k ,
so no value of k will make them equal,

- (b) Let $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation given by $R(\vec{x}) = A\vec{x}$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation given by $S(\vec{x}) = B\vec{x}$, where $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, given by $T(\vec{x}) = S(R(\vec{x}))$. Show all your work!

$$T(\vec{x}) = S(R(\vec{x})) = \underbrace{B(A\vec{x})}_{A\vec{x}} = (BA)\vec{x}$$

The standard matrix of T is

$$BA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix},$$

(c) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \end{pmatrix}$ an arbitrary 2×3 matrix.

Explain why every column of the 3×3 matrix $C := AB$ must belong to the plane $x_1 - 2x_2 + x_3 = 0$ in \mathbb{R}^3 .

$$\vec{a}_1 = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \text{ and } \vec{a}_2 = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

The two columns of A belong to this plane (just plug into the equation of the plane). The plane passes through the origin, hence every linear combination of the columns of A belongs to this plane. More explicitly, the equation of the plane can be written in the form

$$(1 \ -2 \ 1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \quad \text{and}$$

$$(1 \ -2 \ 1)(c_1 \vec{a}_1 + c_2 \vec{a}_2) = c_1 (1 \underbrace{-2 \ 1}_{0}) \vec{a}_1 + c_2 (1 \underbrace{-2 \ 1}_{0}) \vec{a}_2 = 0,$$

Now every column of $AB = (Ab_1 \ Ab_2 \ Ab_3)$ is a linear combination of the columns of A and hence belongs to the plane.

5. (20 points) Determine if the statement is true or false. Justify your answer! (credit will be given only if a valid justification is provided).

- (a) If A is a 4×3 matrix, then there must be a vector \vec{b} in \mathbb{R}^4 , such that the equation $A\vec{x} = \vec{b}$ is inconsistent.

True,

A can not have a pivot position in every row, as there are more rows than columns,

(Recall that $A\vec{x} = \vec{b}$ is consistent for all \vec{b} if and only if A has a pivot position in every row),

- (b) If the columns of a square $n \times n$ matrix are linearly independent, then they span the whole of \mathbb{R}^n .

True.

columns are linearly independent $\Rightarrow \left\{ \begin{array}{l} \text{the matrix has a} \\ \text{pivot position in every} \\ \text{column} \end{array} \right\} \Rightarrow$

\Rightarrow the matrix has a pivot position in every row $\Rightarrow \left\{ \begin{array}{l} \text{The columns} \\ \text{span the} \\ \text{whole of} \\ \mathbb{R}^n \end{array} \right\}$

\uparrow

Since the matrix is square

- (c) There exist three vectors v_1, v_2, v_3 in \mathbb{R}^5 , such that the set $\{v_1 + v_2, v_1 + v_3, 2v_1 + v_2 + v_3\}$ is linearly independent.

$\underbrace{}_{w_1}$

w_3

$\underbrace{}_{w_2}$

FALSE,

The third vector w_3 is the sum of the first two

$$w_3 = w_1 + w_2.$$

Hence, $-w_1 - w_2 + w_3 = 0$ and the coefficients $-1, -1, 1$ are not all zero.