Math 235 Final Exam Fall 2006

1. (15 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & -3 & 4 & -1 & 9 \\ 2 & -6 & 6 & 1 & 10 \\ 3 & -9 & 6 & 6 & 3 \\ 3 & -9 & 4 & 9 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -3 & 0 & 5 & 2 \\ 0 & 0 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   a) Find a basis for the null space $Null(A)$ of A.

   b) Find a basis for the column space of A.

   c) Find a basis for the row space of A.

2. (15 points)

   (a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} -5 & 3 & 6 \\ -6 & 4 & 6 \\ 0 & 0 & 1 \end{pmatrix}$ is $-(\lambda - 1)^2(\lambda + 2)$.

   (b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

   (c) Find an invertible matrix P and a diagonal matrix D such that the matrix A above satisfies
$$P^{-1}AP = D$$

3. (15 point) i) Let A be a 6×10 matrix (6 rows and 10 columns). Denote the dimension of the null space of A by k.

   (a) Express the rank of A in terms of k.
   rank(A) = _________.

   (b) Express the dimension of the column space of A in terms of k.
   dim (Col(A)) = _________.

   ii) Let A be a 3×2 matrix and B a 2×3 matrix. Their product AB is thus a 3×3 matrix.

   (a) Show that each column of AB is a linear combination of the columns of A. Conclude, that the column space Col(AB) is a subspace of Col(A).

   (b) Show that Null(B) is a subspace of Null(AB).

   (c) Use your work above to show that $\text{rank}(AB) \leq \min\{\text{rank}(A),\ \text{rank}(B)\}$.

   (d) Can the product AB be invertible? Justify your answer!

4. (15 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix}$.

(a) The eigenvalue of v_1 is ______

The eigenvalue of v_2 is ______

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

(c) Compute $A^{100}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(d) As n gets larger, the vector $A^n\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ approaches _____. Justify your answer.

5. (15 points) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$.

(a) Find the projection of $b = \begin{bmatrix} -1 \\ 11 \\ 3 \end{bmatrix}$ to the plane col(A) spanned by the columns of A.

(b) Find the distance from b to col(A).

(c) Find a least square solution of the equation $Ax = b$. I.e., find a vector x in $\mathbb{R}^2$, for which the distance $\|Ax - b\|$ from Ax to b is minimal.

6. (15 points) Let $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$.

(a) Write v as a sum $v = \hat{v} + u_2$ of a vector $\hat{v}$ parallel to u_1 and a vector u_2 orthogonal to u_1.

(b) Find the distance from v to the line spanned by u_1.

(c) Find an orthogonal basis for the plane W in $\mathbb{R}^3$ spanned by u_1 and v.

(d) Find a vector u_3, such that the above two vectors u_1, u_2 combine with u_3 to give an orthogonal basis $\{u_1, u_2, u_3\}$ of $\mathbb{R}^3$.

7. (10 points)

(a) Find the inverse P^{-1} of the matrix $P = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$.

(b) Denote the j-th column of P by p_j. Let A be the 3×3 matrix satisfying

$$Ap_1 = 2p_1, \quad Ap_2 = -p_2, \quad Ap_3 = p_3.$$

Calculate A. (**Check** that the A you found satisfies the three equations!). Hint: First find $P^{-1}AP$.