Math 235 Final Exam Fall 2001

Name: MY SOLUTION

1. (16 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & -2 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 2 & -2 & 1 \end{pmatrix} \qquad B = \begin{array}{c} \begin{array}{cccccc} & & x_3 & & x_5 & x_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

6 pts

a) Find a basis for the null space $Null(A)$ of A. x_3, x_5, x_6 are free

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 + x_6 \\ -2x_3 + x_6 \\ x_3 \\ x_5 - x_6 \\ x_5 \\ x_6 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Basis

5 pts

b) Find a basis for the column space of A. a_1, a_2, a_4,

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

5 pts

c) Find a basis for the row space of A. The non-zero rows of B

2. (16 points)

5 pts

(a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} 6 & 0 & -4 \\ 0 & 1 & 0 \\ 8 & 0 & -6 \end{pmatrix}$ is $-(\lambda-1)(\lambda+2)(\lambda-2)$.

$$\det |A-\lambda I| = \begin{vmatrix} 6-\lambda & 0 & -4 \\ 0 & 1-\lambda & 0 \\ 8 & 0 & -6-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 6-\lambda & -4 \\ 8 & -6-\lambda \end{vmatrix} = (1-\lambda)\underbrace{\left[(\lambda^2-36)+32\right]}_{\underbrace{\lambda^2-4}_{(\lambda-2)(\lambda+2)}}$$

$$= -(\lambda-1)(\lambda+2)(\lambda-2)$$

6 pts

(b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

$\underline{\lambda_1 = 1:}$ $A - I = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 0 & 0 \\ 8 & 0 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad Null(A-I) = Sp\left\{\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1}\right\}$

$\underline{\lambda_2 = 2:}$ $A - 2I = \begin{bmatrix} 4 & 0 & -4 \\ 0 & -1 & 0 \\ 8 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad x_3 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$

$\underline{\lambda_3 = -2:}$ $A + 2I = \begin{bmatrix} 8 & 0 & -4 \\ 0 & 3 & 0 \\ 8 & 0 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ & & \end{bmatrix}$ $\quad x_3 \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}}_{v_3}$

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5 pts

(c) Find an invertible matrix P and a diagonal matrix D such that the matrix A above satisfies

$$P^{-1}AP = D$$

$$P = \begin{bmatrix} 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & -2 \end{bmatrix}$$

3. (4 point) Let A be a 6×10 matrix (6 rows and 10 columns). Denote the dimension of the column space of A by r.

2 pts

(a) The dimension r of the column space must be in the range $\underline{0} \leq r \leq \underline{6}$.

1 pt

(b) Express the dimension of the null space of A in terms of r. $\dim \mathrm{Null}(A) = \underline{10 - r}$.

1 pt

(c) Express the dimension of the row space of A in terms of r. $\dim \mathrm{Row}(A) = \underline{r}$.

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4. (16 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .7 & .3 \\ .3 & .7 \end{pmatrix}$.

4 pts (a) The eigenvalue of v_1 is 1

The eigenvalue of v_2 is .4

4 pts (b) Find the coordinates of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \end{array}\right]$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

4 pts (c) Compute $A^{100} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$A^{100} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - (0.4)^{100} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

4 pts (d) As n gets larger, the vector $A^n \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ approaches $\frac{3}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Justify your answer.

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5. (16 points) Let W be the plane in $\mathbb{R}^3$ spanned by $u_1 = \begin{bmatrix} 4 \\ -1 \\ -8 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -7 \\ 4 \\ -4 \end{bmatrix}$.

$u_1 \cdot u_2 = -28 - 4 + 32 = 0$

4 pts

(a) Find the projection $\mathrm{Proj}_W(v)$ of $v = \begin{bmatrix} 8 \\ 7 \\ -7 \end{bmatrix}$ to W.

$$\hat{v} = \frac{v \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{v \cdot u_2}{u_2 \cdot u_2} \cdot u_2 = \frac{\overbrace{32 - 7 + 56}^{81}}{\underbrace{16 + 1 + 64}_{81}} u_1 + \frac{\overbrace{-56 + \overbrace{28}^{56} + 28}^{0}}{*} \cdot u_2 = u_1 = \begin{bmatrix} 4 \\ -1 \\ -8 \end{bmatrix}$$

4 pts

(b) Find the distance from v to W.

$$\text{distance} = \| v - \hat{v} \| = \left\| \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix} \right\| = \sqrt{16 + 64 + 1} = \sqrt{81} = 9$$

4 pts

(c) Set $u_3 := \overbrace{v - \mathrm{Proj}_W(v)}^{\begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}}$ and let U be the 3×3 matrix with columns u_1, u_2, and u_3. Show that $\frac{1}{9}U$ is an **orthogonal** matrix.

$$\underbrace{\frac{1}{9}\begin{bmatrix} 4 & -7 & 4 \\ -1 & 4 & 8 \\ -8 & -4 & 1 \end{bmatrix}}_{U} \cdot \underbrace{\frac{1}{9}\begin{bmatrix} 4 & -1 & -8 \\ -7 & 4 & -4 \\ 4 & 8 & 1 \end{bmatrix}}_{U^T} = \frac{1}{81}\begin{bmatrix} \overbrace{(16+49+16)}^{81} & \overbrace{(-4-28-32)}^{0} & \overbrace{(-32+28+4)}^{0} \\ & \overbrace{(1+16+64)}^{81} & \overbrace{(8-16+8)}^{0} \\ & & \underbrace{(64+16+1)}_{0} \end{bmatrix}$$

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4 pts

(d) Find the distance, from the vector $\left(\frac{1}{9}U^T\right)v$ to the plane in $\mathbb{R}^3$ spanned by the vactors $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$, without any further calculations. Explain your answer! *Hint: where does $\frac{1}{9}U$ take the three vectors above?*

$\frac{1}{9}U$ takes $\begin{bmatrix}1\\0\\0\end{bmatrix}$ to $\frac{1}{9}u_1$

" " $\begin{bmatrix}0\\1\\0\end{bmatrix}$ to $\frac{1}{9}u_2$

" " $\left(\frac{1}{9}U^T\right)v$ to v

$\frac{1}{9}U$ preserves distances. Thus, the answer is equal to the distance from v to the plane spanned by u_1, u_2, which is the answer in part b. $\boxed{9}$

4 bonus pts

(e) $\frac{1}{9}U$ is the matrix of a rotation of $\mathbb{R}^3$ about a line L through the origin (you may assume this fact). Find a vector w which spans the line L (the axis of rotation).

$$\begin{bmatrix} \frac{4}{9}-1 & -\frac{7}{9} & \frac{4}{9} \\ -\frac{1}{9} & \frac{4}{9}-1 & \frac{8}{9} \\ -\frac{8}{9} & -\frac{4}{9} & \frac{1}{9}-1 \end{bmatrix} \sim \begin{bmatrix} -5 & -7 & 4 \\ -1 & -5 & 8 \\ -8 & -4 & -8 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 5 & -8 \\ 0 & 18 & -36 \\ 0 & 36 & -72 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & -8 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}_{w}$$

eigenvector with eigenvalue 1

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6. (16 points) Let W be the plane in $\mathbb{R}^3$ spanned by $a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $a_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

(a) Find the projection of a_2 to the line spanned by a_1.

$$\text{Proj}_{a_1}(a_2) = \frac{a_1 \cdot a_2}{a_1 \cdot a_1} \cdot a_1 = \frac{3}{3} \cdot a_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3 pts

(b) Find the distance from a_2 to the line spanned by a_1.

$$\text{distance} = \| a_2 - \text{Proj}_{a_1}(a_2) \| = \left\| \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\| = \sqrt{2}$$

3 pts

4 pts

(c) Use your calculations in parts 6a and 6b to show that the vectors $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ form an orthogonal basis of the plane W given above.

$u_1 = a_1$

$u_2 = a_2 - \text{Proj}_{a_1}(a_2)$, and is thus both orthogonal to u_1 and is a linear comb of a_1, a_2.

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3 pts

(d) Find the projection of $b = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ to W.

$$\hat{b} = \frac{b \cdot u_1}{u_1 \cdot u_1} \cdot u_1 + \frac{b \cdot u_2}{u_2 \cdot u_2} \cdot u_2 = \frac{6}{3} \cdot u_1 + \frac{2}{2} u_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

3 pts

(e) Find a least square solution of the equation $Ax = b$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ is the 3×2 matrix with columns a_1 and a_2. I.e., find a vector x in $\mathbb{R}^2$, for which the distance $\|Ax - b\|$ from Ax to b is minimal.

Method 1:

Solve $A\vec{x} = \hat{b} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

So, $x_1 = 1$
$x_2 = 1$

Method 2: $A^T A \vec{x} = A^T \hat{b} = \underbrace{A^T b}$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 5 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

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7. (16 points)

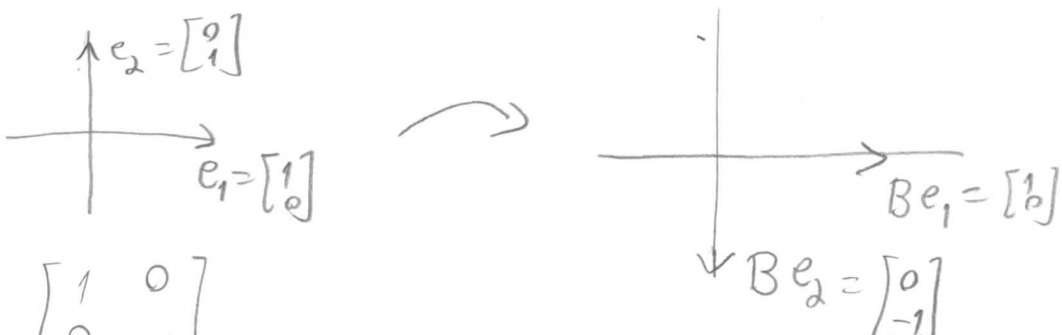
5 pts

(a) Find the matrix A of the rotation of $\mathbb{R}^2$ an angle of $\frac{\pi}{4}$ radians (45°) counter-clockwise.

$$\begin{pmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

5 pts

(b) Find the matrix B of the reflection of the plane about the line $x_2 = 0$ (the x_1 coordinate line).



$$B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

3 pts

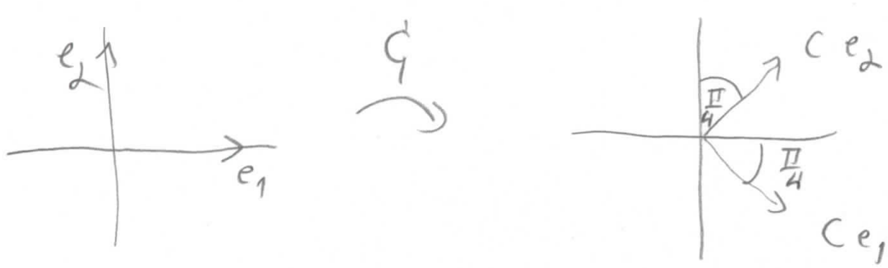
(c) Compute $C = B^{-1}AB$.

$B^{-1} = B$

$$B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

3 pts

(d) Show that C is the matrix of a rotation and find the angle of rotation.



Rotation by angle $\frac{\pi}{4}$ clockwise

($-\frac{\pi}{4}$ counterclockwise)