Math 235 Final Exam Fall 2015

1. (20 points)

   (a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} 5 & 0 & 4 \\ -2 & 3 & -4 \\ 2 & 0 & 7 \end{pmatrix}$ is $-(\lambda - 3)^2(\lambda - 9)$.

   (b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of $A$.

   (c) Find an invertible matrix $P$ and a diagonal matrix $D$ such that the matrix $A$ above satisfies
$$P^{-1}AP = D$$

   (d) Let $B$ be a $5 \times 5$ matrix with characteristic polynomial $-(\lambda - 1)^2(\lambda - 2)(\lambda - 3)(\lambda - 4)$. Assume that the rank of $B - I$ is 3. Is $B$ necessarily diagonalizable? Justify your answer.

2. (20 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .6 & .4 \\ .3 & .7 \end{pmatrix}$.

   (a) The eigenvalue of $v_1$ is ______

   The eigenvalue of $v_2$ is ______

   (b) Find the coordinates of $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

   (c) Compute $A^{20}\begin{pmatrix} 1 \\ 8 \end{pmatrix}$.

   (d) As $n$ gets larger, the vector $A^n\begin{pmatrix} 1 \\ 8 \end{pmatrix}$ approaches _____. Justify your answer.

   (e) Let $B$ be an invertible $n \times n$ matrix and $v$ an eigenvector of $B$ with eigenvalue 5. Show that $v$ is an eigenvector of the inverse matrix $B^{-1}$ as well and compute its eigenvalue.

3. (20 points) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$.

   (a) Find the projection of $b$ to the plane $\text{Col}(A)$ spanned by the columns of $A$.

   (b) Find the distance from $b$ to $\text{Col}(A)$.

   (c) Find a vector $x$ in $\mathbb{R}^2$, for which the distance $\|Ax - b\|$ from $Ax$ to $b$ is equal to the distance from $b$ to $\text{Col}(A)$. Hint: The vector $Ax$ is in $\text{Col}(A)$ for every vector $x$.

4. (20 points) Consider the following orthogonal basis of $\mathbb{R}^3$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

(a) Find the coordinates of the vector $b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in the above basis.

(b) Normalize the above basis $\{v_1, v_2, v_3\}$ to an orthonormal basis $\{u_1, u_2, u_3\}$.

(c) Let $A$ be an $n \times n$ orthogonal matrix and $v$ an eigenvector of $A$ in $\mathbb{R}^n$. Show that the eigenvalue of $v$ is either 1 or $-1$. Hint: Consider the length of $Av$.

5. (20 points)

(a) If the null space of an $8 \times 5$ matrix is 2 dimensional, what is the dimension of the row space of $A$? Justify your answer.

(b) Show that the first three Laguerre polynomials $\{1, 1-t, 2-4t+t^2\}$ form a basis of $\mathbb{P}_2$. Explain, **in complete sentences**, why it is linearly independent and why it spans $\mathbb{P}_2$.

(c) Let $\mathcal{B}$ be the basis $\left\{ \begin{pmatrix} 1 \\ -4 \end{pmatrix}, \begin{pmatrix} -2 \\ 9 \end{pmatrix} \right\}$ of $\mathbb{R}^2$ and $[\ ]_{\mathcal{B}} : \mathbb{R}^2 \to \mathbb{R}^2$ the coordinate linear transformation sending a vector $v$ to its coordinate vector $[v]_{\mathcal{B}}$ relative to the basis $\mathcal{B}$. Find the matrix $A$ of the linear transformation $[\ ]_{\mathcal{B}}$. **Justify your answer!** Hint: Multiplication by $A$ should transform a vector $v$ into its coordinate vector $[v]_{\mathcal{B}}$.