Math 235 Section 1 Final Exam Fall 2008

Justify all your answers. Show all your work!!!

1. (15 points) The matrices A and B below are row equivalent (you do **not** need to check this fact). $A = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & -2 \\ 3 & 0 & 6 & 0 & -3 & 3 \\ 0 & 1 & 4 & 2 & 1 & -2 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 0 & 2 & 0 & -1 & 1 \\ 0 & 1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

   a) Find a basis for $\ker(A)$.

   b) Find a basis for image(A).

   c) What are all the possible ranks of a 6×6 matrix C, given that C satisfies $AC = 0$ (the product is the zero matrix), where A is the matrix given above? **Justify** your answer! Hint: Additional calculations are not needed; use instead the Rank Nullity Theorem.

2. (15 points) Let $\vec{v} := \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Recall that the reflection $T : \mathbb{R}^2 \to \mathbb{R}^2$, with respect to the line spanned by $\vec{v}$, is given by $T(\vec{x}) = 2\left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} - \vec{x}$.

   (a) Show that the matrix of T, with respect to the standard basis, is $A = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$. Show all your work!

   (b) Let $\vec{w} := \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $S(\vec{x}) = 2\left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w} - \vec{x}$ the reflection of $\mathbb{R}^2$ with respect to the line spanned by $\vec{w}$. One can similarly show that the matrix of S, with respect to the standard basis, is $B = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$. You may assume this equality. Use the matrices of T and S in order to find the matrix of the composition $ST : \mathbb{R}^2 \to \mathbb{R}^2$, taking $\vec{x}$ to $S(T(\vec{x}))$. Show all your work!

   (c) Determine if the linear transformation $ST : \mathbb{R}^2 \to \mathbb{R}^2$ is a reflection, a rotation, or neither. Justify your answer!

3. (20 points) Consider the matrix $A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

   (a) Show that the characteristic polynomial of A is $-(\lambda - 1)^2(\lambda - 5)$. You are *required* to calculate the determinant of the suitable 3×3 matrix B, with entries depending on λ, using the affect of elementary row operations on the determinant (credit will not be given for a calculation using another method!). Perform first the following four elementary row operations on B: i) Interchange the first and third row. ii) Add a multiple of the new first row to the new second row, to annihilate the $(2, 1)$ entry. iii) Add a multiple of the new first row to the new third row, to annihilate the $(3, 1)$ entry. iv) Multiply the new second row by the appropriate factor, so that its leading entry becomes 1. In each step clearly state in words what is the affect of the elementary row operation on the determinant.

   (b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

   (c) Find an invertible matrix S and a diagonal matrix D such that the matrix A above satisfies $S^{-1}AS = D$.

4. (20 points) A kid has two favorite snacks, snack 1 (chocolate) and snack 2 (ice cream). He eats the snack of his choice each evening. The (i, j) entry $a_{i,j}$, of the matrix $A = \begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$, is the probability of him choosing snack i the next day, if on a given day he chose snack j. If, for example, the kid chose snack 1 today, then the probability of him choosing snack 1 again tomorrow is $a_{1,1} = 0.8$ (which stands for 80%), and the probability of him choosing snack 2 tomorrow is $a_{2,1} = 0.2$. It can be shown (and you may assume it) that *the (i, j) entry of the n-power A^n is the probability of him choosing snack i after n days, if on a given day he chose snack j.*

   The vectors $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ are eigenvectors of the matrix A.

   (a) The eigenvalue of v_1 is ______. The eigenvalue of v_2 is ______.

   (b) Set $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Find the coordinate vectors $[e_1]_\beta$ and $[e_2]_\beta$ of e_1 and e_2 in the basis $\beta := \{v_1, v_2\}$. (You should get that the second cordinate of $[e_1]_\beta$ is equal to the second cordinate of $[e_2]_\beta$).

   (c) Compute $A^{365}e_i$, for $i = 1, 2$.

   (d) Show that as n gets larger, the two vectors $A^n e_1$ and $A^n e_2$ approach the same vector $\lim_{n\to\infty}(A^n e_1) = \lim_{n\to\infty}(A^n e_2)$. (Calculate this vector in each case).

   (e) Use the above highlighted interpretation of the entries of A^n, and your work above, in order to explain the following statement: *Todays snack choice has diminishing affect on the probability of the kid choosing chocolate or ice cream in n days, as n gets larger.*

5. (20 points) Let $C^\infty(\mathbb{R})$ be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$, having derivatives of all orders. Denote by V the subspace of $C^\infty(\mathbb{R})$ spanned by the functions $f_1(x) = e^x$, $f_2(x) = e^{2x}$, and $f_3(x) = e^{3x}$. Let $T : V \to \mathbb{R}^3$ be the transformation given by $T(f) := \left(f(0), f'(0), f''(0) \right)$.

   (a) Show that the transformation T is linear. In other words, verify the following identities, for any two elements f, g of V, and for every scalar k.

      i. $T(f + g) = T(f) + T(g)$.

      ii. $T(kf) = kT(f)$.

   (b) Show that the subset $\{T(f_1), T(f_2), T(f_3)\}$ of $\mathbb{R}^3$ is linearly independent. Hint: Recall that the chain rule yields $(e^{2x})' = 2e^{2x}$, $(e^{2x})'' = 2^2e^{2x}$, and so $f_2''(0) = 4$.

   (c) Show that $\mathrm{im}(T)$ is the whole of $\mathbb{R}^3$.

   (d) Show the the subset $\{e^x, e^{2x}, e^{3x}\}$ of V is linearly independent. Start your answer with the definition of a linear independent subset of $C^\infty(\mathbb{R})$.

   (e) Show that $T : V \to \mathbb{R}^3$ is an isomorphism.

6. (15 points) Let V be the plane in $\mathbb{R}^3$ spanned by $v_1 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $v_2 := \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$.

   (a) Find the orthogonal projection $\mathrm{proj}_V(w)$ of $w = \begin{pmatrix} 12 \\ -2 \\ -2 \end{pmatrix}$ into V.

   (b) Write w as a sum of a vector in V and a vector orthogonal to V.

   (c) Find the distance from w to V (i.e., to the vector in V closest to w).