Math 235 Section 1 Midterm 2 Fall 2000

1. (18 points) You are given below the matrix A together with its row reduced echelon form B (you need *not* verify that B is indeed the reduced echelon form of A)

$$A = \begin{pmatrix} 1 & -1 & -3 & -3 & 0 & -3 \\ 1 & 0 & 2 & 3 & 0 & 4 \\ 2 & 0 & 4 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 & 3 & 0 & 4 \\ 0 & 1 & 5 & 6 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) Find a spanning set for (i.e., a set of vectors which spans) the null space $Null(A)$ of A.

b) Are the column spaces col(A) and col(B) equal? **Justify** your answer carefully! (either explain why they are equal, or find a vector in one that does not belong to the other).

2. (16 points) Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the 2×2 identity matrix and $A = \begin{pmatrix} 0 & 6 \\ 1 & 5 \end{pmatrix}$.

a) Show that the matrix $A - 5I$ is *invertible*. Note that $A - 5I$ is simply $\begin{pmatrix} -5 & 6 \\ 1 & 0 \end{pmatrix}$.

b) Express the determinant of the 2×2 matrix $A - tI$ in terms of the scalar parameter t.

c) Show that there are precisely two values of t for which the matrix $A - tI$ is *not* invertible. Find these values of t.

3. (18 points) Determine if the following set in $\mathbb{R}^n$ is a subspace. If it is not, find a property in the definition of a subspace which this set violates. If it is a subspace, find a matrix A such that this set is either $Null(A)$ or $Col(A)$.

(a) $$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x, y, z \text{ are real numbers satisfying } \begin{array}{rcl} 2x + 3y + 4z & = & 5 \\ x + z & = & 6 \end{array} \right\}$$

(b) $$\left\{ \begin{bmatrix} a + 2b \\ b - 2c \\ 2c + 5d \\ c - a \end{bmatrix} \text{ such that } a, b, c, d \text{ are arbitrary real numbers} \right\}$$

(c) $$\left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \text{ such that } x, y, z, w \text{ are real numbers satisfying } \begin{array}{rcl} z & = & x + y \\ w & = & x - y \end{array} \right\}$$

4. (16 points) a) Compute the volume of the parallelepiped in $\mathbb{R}^3$ with vertices $\vec{0}$, v_1, v_2, v_3, $v_1 + v_2$, $v_1 + v_3$, $v_2 + v_3$, $v_1 + v_2 + v_3$ where

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$$

b) Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ sending a vector $\vec{x}$ to $A\vec{x}$, where A is the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{bmatrix}$. Compute the volume of the parallelepiped obtained as the image of the one in part (a) under the transformation T. Express your answer in terms of a, b and c. *Note:* The image is the parallelepiped with vertices $\vec{0}$, Av_1, Av_2, Av_3, $A(v_1 + v_2)$, $A(v_1 + v_3)$, $A(v_2 + v_3)$, $A(v_1 + v_2 + v_3)$.

5. a) (6 points) Let A, B, and C be 3×3 matrices satisfying the equation

$$B^2 + 2B = ACA^{-1}$$

with A invertible. Solve for C in terms of A and B.

b) (10 points) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 3 \end{pmatrix}$ Compute its inverse A^{-1}. (**Check** that $AA^{-1} = I$.)

c) (6 points) Compute the $(2, 3)$ entry of ABA^{-1} if A is given in part (b) and $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$

6. (10 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree ≤ 2. Recall that a vector in $\mathbb{P}_2$ is a polynomial $p(t)$ of the form $p(t) = a_0 + a_1 t + a_2 t^2$ where the coefficients a_0, a_1, a_2 are arbitrary real numbers.

   (a) Show that the subset H of $\mathbb{P}_2$ of polynomials $p(t)$ of degree ≤ 2 which in addition satisfy

   $$p(0) = 0 \quad \text{and} \quad p(1) = 0$$

   is a *subspace* of $\mathbb{P}_2$. (The straightforward answer would include the definition of a subspace and a verification that H satisfies all the properties.)

   (b) Find a spanning set for H. **Explain** why the set you found spans H.