Math 235 Final Exam Fall 2000

1. (15 points) The matrices A and B below are row equivalent (you do **not** need to check this fact).

$$A = \begin{pmatrix} 1 & -2 & 1 & 1 & 1 \\ 2 & -4 & 0 & 1 & 3 \\ -3 & 6 & 1 & 1 & -3 \\ -1 & 2 & 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a) What is the rank of A?

b) Find a basis for the null space $Null(A)$ of A.

c) Find a basis for the column space of A.

d) Find a basis for the row space of A.

2. (6 points) The system $A\vec{x} = 0$ has a 2-dimensional space of solutions and the size of the matrix A is 6×5. What is the dimension of (a) the Null space of A? (b) the Column space of A? (c) the Row space of A? **Justify your answers!**

3. (15 points)

   (a) Show that the characteristic polynomial of the matrix $A = \begin{pmatrix} -1 & -2 & -4 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{pmatrix}$ is $-(\lambda - 1)(\lambda + 1)(\lambda - 2)$.

   (b) Find a basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

   (c) Find an invertible matrix P and a diagonal matrix D such that the matrix A above satisfies
   $$P^{-1}AP = D$$

4. (12 points) Determine for which of the following matrices A below there exists an invertible matrix P (with real entries) such that $P^{-1}AP$ is a diagonal matrix. You do **not** need to find P. **Justify your answers!**

   (a) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

5. (20 points) Let W be the plane in $\mathbb{R}^3$ spanned by $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

Note: Parts 5a, 5b are mutually independent and are not needed for doing parts 5c, 5d, 5e.

(a) Find the distance between the two points v_1 and v_2 in $\mathbb{R}^3$.

(b) Find a vector of length 1 which is orthogonal to W.

(c) Find the projection of v_2 to the line spanned by v_1.

(d) Write v_2 as the sum of a vector parallel to v_1 and a vector orthogonal to v_1.

(e) Find an orthogonal basis for W.

(f) Find an *orthogonal* 3×3 matrix U, such that the corresponding linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ takes the x_1 axis to the line spanned by v_1 and the x_1, x_2 coordinate plan to W. *Hint: Use parts 5b and 5d.*

6. (16 points) Let W be the plane in $\mathbb{R}^3$ spanned by $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

(a) Find the projection of $b = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ to W.

(b) Find the distance from b to W.

(c) Find a least square solution of the equation $Ax = b$, where $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}$ is the 3×2 matrix with columns u_1 and $u_1 + u_2$. I.e., find a vector x in $\mathbb{R}^2$ which minimizes the length $\|Ax - b\|$.

(d) Find the coefficients c_0, c_1 of the line $y(x) = c_0 + c_1 x$ which best fits the three points $(x_1, y_1) = (1, 2)$, $(x_2, y_2) = (-2, 1)$, $(x_3, y_3) = (1, -2)$ in the x, y plane. The line should minimize the sum $\sum_{i=1}^{3} [y(x_i) - y_i]^2$. **Justify your answer!**

7. (16 points) The vectors $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are eigenvectors of the matrix $A = \begin{pmatrix} .4 & .6 \\ .6 & .4 \end{pmatrix}$.

(a) The eigenvalue of v_1 is ______

The eigenvalue of v_2 is ______

(b) Find the coordinates of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the basis $\{v_1, v_2\}$.

(c) Compute $A^{50} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(d) As n gets larger, the vector $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ approaches _____. Justify your answer.