DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MATH 132 SPRING 2004
EXAM 2 — SOLUTIONS

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1. (15) For each of the following improper integrals, determine whether the integral converges or diverges. Evaluate it, showing all your algebraic steps, if it is convergent. Otherwise, explain why it is divergent.

a) $\displaystyle\int_0^4 \frac{1}{(x-2)^3}\,dx$

Solution: The integrand has a vertical asymptote at $x = 2$ so we need to break up the integral and consider appropriate limits:

$$\int_0^4 \frac{1}{(x-2)^3}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{(x-2)^3}\,dx + \lim_{t\to 2^+}\int_t^4 \frac{1}{(x-2)^3}\,dx$$

Consider the first integral for $0 \le t < 2$:

$$\int_0^t \frac{1}{(x-2)^3}\,dx = \left[(-2)\frac{1}{(x-2)^2}\right]_0^t = \frac{-2}{(t-2)^2} + \frac{1}{2}.$$

As $t \to 2^-$, the first fraction $-2/(t-2)^2$ goes to infinity, so the first limit diverges. Thus, the original integral diverges.

b) $\displaystyle\int_1^\infty \frac{1+x^3e^{-x}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \left(\frac{1}{x^2} + xe^{-x}\right)\,dx$

Solution: The integrand has a pole at $x = 0$ which is harmless since the integration only starts at $x = 1$. So we only need to replace the ∞ of the upper limit, as above.

$$\begin{aligned}
\int_1^t \left(\frac{1}{x^2} + xe^{-x}\right)\,dx &= \left[-\frac{1}{x}\right]_1^t + \left[-xe^{-x} - e^{-x}\right]_1^t \text{ (int. by parts)}\\
&= -\frac{1}{t} + \frac{1}{1} + \left(-te^{-t} - e^{-t} - \left(-1e^{-1} - e^{-1}\right)\right)\\
&= -\frac{1}{t} - te^{-t} - e^{-t} + 1 + \frac{2}{e}
\end{aligned}$$

Note that $\lim_{t\to\infty} -1/t = 0$, $\lim_{t\to\infty} e^{-t} = 0$ and $\lim_{t\to\infty}(-te^{-t}) = 0$ (using L'Hôpital), so that $\displaystyle\int_1^\infty \frac{1+x^3e^{-x}}{x^2}\,dx = 1 + \frac{2}{e}$.

2. (15) Find the equation of the line tangent to the parametrized curve

$$\begin{cases} x(t) &= t^2+1 \\ y(t) &= \sqrt{t+1} \end{cases}$$

at $t = 3$.

Solution: For the slope, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2\sqrt{t+1}}}{2t} = \frac{1}{4t\sqrt{t+1}} = \frac{1}{24} \text{ at } t = 3.$$

When $t = 3$ we have $(x, y) = (10, 2)$ so the equation of the tangent line is

$$\begin{aligned} (y-2) &= \frac{1}{24}(x-10) \text{ or} \\ y &= \frac{1}{24}x + \frac{19}{12}. \end{aligned}$$

3. (20) a) (7) Eliminate the parameter to find a Cartesian equation for the parametrized curve

$$\begin{cases} x(\theta) &= 4\cos(\theta) \\ y(\theta) &= 3\sin(\theta), \end{cases}$$

$0 \le \theta \le 2\pi$.

Solution: We have the trig identity $\sin^2\theta + \cos^2\theta = 1$. Solving the above equations for the trig functions, we get

$$\cos\theta = \frac{x}{4} \qquad \sin\theta = \frac{y}{3},$$

which we plug into $\sin^2\theta + \cos^2\theta = 1$ to obtain

$$\left(\frac{y}{3}\right)^2 + \left(\frac{x}{4}\right)^2 = 1.$$

b) (6) Sketch the curve in the (x, y) plane, and determine the direction in which the curve is traversed with increasing θ.

Solution: The curve is an ellipse centered at the origin, intersecting the coordinate axes at $(4, 0)$, $(-4, 0)$, $(0, 3)$ and $(0, -3)$. The curve is traversed in the counter-clockwise direction as θ increases from 0 to 2π, starting and ending at $(4, 0)$.

c) (7) Use the parametrization of the curve in part a) to express its arclength as a definite integral. Do **NOT** evaluate the integral.

Solution: The formula for the arclength of a curve given by parametric equations depending on θ is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta,$$

so we need to compute $dx/d\theta$ and $dy/d\theta$:

$$\frac{dx}{d\theta} = -4\sin\theta, \qquad \frac{dy}{d\theta} = 3\sin\theta,$$

which gives

$$L = \int_0^{2\pi} \sqrt{16\sin^2\theta + 9\cos^2\theta}\, d\theta.$$

4. (15) a) (6) Sketch the curve given in polar coordinates by the equation

$$r = \cos(\theta).$$

Solution: The curve is a circle centered at $(1/2, 0)$ with radius $1/2$, as we will see in part c).

b) (7) Find a Cartesian equation for the curve in part a).

Solution: We have

$$x = r \cdot \cos\theta, \tag{1}$$
$$y = r \cdot \sin\theta, \tag{2}$$
$$r^2 = x^2 + y^2. \tag{3}$$

Seeing that the equation of the curve $r = \cos(\theta)$ contains $\cos(\theta)$, we can solve equation (1) for $\cos(\theta)$ and use the equation of the curve:

$$\frac{x}{r} = \cos\theta = r.$$

Multiplying $x/r = r$ by r, we find

$$x = r^2 = x^2 + y^2 \qquad \text{now using equation (3).}$$

In short:

$$x^2 - x + y^2 = 0.$$

c) (2) Show that the curve is a circle and find its center and radius.

Solution: Completing the square for x in the above equation gives:

$$x^2 - x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + y^2 = 0 \qquad \text{or}$$
$$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2.$$

So we see it is a circle centered at $(1/2, 0)$ with radius $1/2$.

5. (15) Find the area enclosed by the curve

$$r = \sin(2\theta), \qquad 0 \le \theta \le 2\pi.$$

Show all your algebraic steps.

Solution:

$$\begin{aligned} A &= \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} \sin^2(2\theta) \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos(4\theta)) \, d\theta = \frac{1}{4} \left[\theta - \frac{\sin(4\theta)}{4} \right]_0^{2\pi} \\ &= \frac{1}{4}(2\pi - 0 - (0 - 0)) = \frac{\pi}{2}. \end{aligned}$$

6. (20) For each of the following sequences determine whether the sequence converges or diverges. If it converges, find the limit, showing your algebraic steps. Otherwise, explain why it diverges.

a) $a_n = \sqrt{\dfrac{2n+3}{3n-1}}, \quad n \ge 1.$

Solution: Clearly,

$$\lim_{n\to\infty} \frac{2n+3}{3n-1} = \frac{2}{3} \qquad \text{(eg. using L'Hôpital)},$$

$$\text{so } \lim_{n\to\infty} \sqrt{\frac{2n+3}{3n-1}} = \sqrt{\frac{2}{3}}$$

and the sequence is convergent.

b) $a_n = \cos(n\pi), \quad n \ge 1.$

Solution: We find $\{a_n\} = \{1, -1, 1, -1, \ldots\}$ so the sequence keeps oscillating between 1 and -1. Therefore, it cannot possibly converge to any of them. This sequence is thus divergent.

c) $a_n = \dfrac{\ln(n)}{\ln(3n)}, \quad n \ge 1.$

As $n \to \infty$, both the numerator and denominator tend towards ∞. We apply L'Hôpital's rule to find the limit:

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(3n)} = \lim_{n\to\infty} \frac{1/n}{1/(3n)\cdot 3} = \lim_{n\to\infty} \frac{1}{1} = 1.$$