MATH 132H FALL 2012 EXAM 2

1. (10 points) Evaluate the integral $\int_0^1 \frac{x}{x^2+5x+6}dx$ using partial fractions. Show all your algebraic steps.

$$\underbrace{x^2+5x+6}_{(x+2)(x+3)}$$

$$\underbrace{\frac{A}{x+2}+\frac{B}{x+3}}=\frac{x}{x^2+5x+6}$$

$$A(x+3)+B(x+2) = x$$

$$A = -2$$

$$-B = -3$$

$$\boxed{A=-2,\ B=3}$$

$$\int_0^1 \frac{x}{x^2+5x+6}\,dx = \int_0^1 \frac{-2}{x+2}+\frac{3}{x+3}\,dx = \Big[-2\ln(x+2)\Big]_0^1 + \Big[3\ln(x+3)\Big]_0^1$$

$$= -2\ln(3)+2\ln(2)+3\ln(4)-3\ln(3)$$

$$= +2\ln(2)-5\ln(3)+3\ln(4).$$

2. a) (8 points) Use the comparison test in order to determine if the following improper integral is convergent or divergent $\int_1^\infty \frac{\ln(x)}{x^2+1}\,dx$. Carefully justify your answer!

Method 1: $0 \le \frac{\ln(x)}{x^2+1} \le \frac{\ln(x)}{x^2}$, for all $x \geq 1$. Now $\int_1^\infty \frac{\ln(x)}{x^2}dx =$

$$= \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x^2}dx = \lim_{t\to\infty} \int_{u(1)=0}^{\ln(t)} \underbrace{\frac{u}{e^u}}_{= u e^{-u}} du = \lim_{s\to\infty} \int_0^s \underbrace{u}\,\underbrace{e^{-u}}_{v'} du =$$

$u = \ln(x)$, $du = \frac{1}{x}dx$, $x = e^u$; $u' = 1$, $v = -e^{-u}$

$$= \lim_{s\to\infty}\left(\underbrace{\left[-ue^{-u}\right]_0^s}_{\frac{s}{e^s}} - \underbrace{\int_0^s -e^{-u}du}_{\underbrace{\left[e^{-u}\right]_0^s}_{e^{-s}-1}} \right) = \lim_{s\to\infty}\left(\frac{s}{e^s} - e^{-s} + 1\right) \overset{(*)}{=} 1 < \infty.$$

$$\boxed{\text{For } (*) \text{ we used L'Hopital: } \lim_{s\to\infty} \frac{s}{e^s} \overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{L'Hop}}{=}} \lim_{s\to\infty}\frac{1}{e^s} = 0.}$$

We see that $\int_1^\infty \frac{\ln(x)}{x^2}dx$ is convergent.

Hence, so is $\int_1^\infty \frac{\ln(x)}{x^2+1}dx$, by the comparison test.

b) (8 points) Evaluate the improper integral showing all your algebraic steps

$$\int_0^1 \frac{e^{(-1/x)}}{x^3}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{e^{-1/X}}{X^3}\,dX =$$

$u = \frac{1}{X}$

$du = -\frac{1}{X^2}\,dX$

$$= \lim_{t\to 0^+} \int_{\frac{1}{t}}^{1} -u e^{-u}\,du = \lim_{s\to\infty} \int_1^s \underbrace{u}_{u}\,\underbrace{e^{-u}}_{v'}\,du =$$

$s = \frac{1}{t}$

$u' = 1 \quad v = -e^{-u}$

$$\lim_{s\to\infty} \Big[-u e^{-u}\Big]_1^s - \underbrace{\int_1^s -e^{-u}\,du}_{\left[e^{-u}\right]_1^s} = \lim_{s\to\infty} \left(-\frac{s}{e^s} + e^{-1}\right) - \left(e^{-s} - e^{-1}\right) =$$

L'Hop $\left[\frac{\infty}{\infty}\right]$: $\lim_{s\to\infty} \frac{1}{e^s} = 0$; $e^{-s} \to 0$

$$= \frac{2}{e}.$$

3. (14 points) For each of the following **sequences** (not series) determine whether the sequence converges or diverges. If it converges, find the limit, showing all your algebraic steps. Otherwise, explain why it diverges.

a) $a_n = \sqrt{n+2} - \sqrt{n}, \quad n \geq 1.$
Hint: Use the identity $(a-b)(a+b) = a^2 - b^2$.

$$a_n = \frac{(\sqrt{n+2} - \sqrt{n})(\sqrt{n+2} + \sqrt{n})}{\sqrt{n+2} + \sqrt{n}} = \frac{(n+2) - n}{\sqrt{n+2} + \sqrt{n}} =$$

$$= \frac{2}{\sqrt{n+2} + \sqrt{n}} = \frac{2/\sqrt{n}}{\sqrt{\frac{n+2}{n}} + 1} \xrightarrow[n\to\infty]{} \frac{0}{2} = 0$$

b) $a_n = (n^2+3)^{1/n}, \quad n \geq 1.$

$$a_n = e^{\ln((n^2+3)^{1/n})} = e^{\frac{1}{n}\ln(n^2+3)}$$

e^x is continuous at every x,

$$\lim_{n\to\infty} \frac{\ln(n^2+3)}{n} = \lim_{x\to\infty} \frac{\ln(x^2+3)}{x} \underset{L'Hop}{\overset{[\frac{\infty}{\infty}]}{=}} \lim_{x\to\infty} \frac{\left[\frac{1}{(x^2+3)} \cdot 2x\right]}{1} =$$

$$= \lim_{x\to\infty} \frac{(2/x)}{1+\frac{3}{x^2}} = \frac{0}{1} = 0,$$

Hence $\lim_{n\to\infty} a_n = e^0 = 1,$

4. (14 points)

9

(a) Find the values of x, for which the series $\sum_{n=0}^{\infty} \frac{(2x-3)^n}{9^n}$ converges. State the convergence test you use and explain why its hypotheses are satisfied.

The series is a geometric series

$$\sum a r^n$$

with $r = \frac{2x-3}{9}$. It converges for $|r| < 1$,

$$\left|\frac{2x-3}{9}\right| < 1$$

$$|2x-3| < 9$$

$$-9 < 2x-3 < 9$$

$$-6 < 2x < 12$$

$$-3 < x < 6$$

5

(b) Find the sum of the series for those values of x. Simplify your answer.

If $-3 < x < 6$, the sum of $\sum_{n=0}^{\infty} a r^n$ is $S = \frac{a}{1-r}$

In our case, $a = 1$,

$$s(x) = \frac{1}{1-\left(\frac{2x-3}{9}\right)} = \frac{9}{9-(2x-3)} = \boxed{\frac{9}{12-2x}}.$$

5. (14 points) Let s be the sum of the series $\sum_{n=1}^{\infty} \frac{4}{n^5}$ and s_n the n-th partial sum.

Find the **minimal** number n of terms of the series, for which we know that $s - s_n \leq 10^{-8}$, by the error estimate of the integral test. Justify your answer, showing all your algebraic steps.

The seq $\frac{4}{n^5}$ is positive, decreasing, and $\int_1^{\infty} 4/x^5 \, dx$ converges.
The error estimate for the integral test is

$$\boxed{s - s_n \leq \int_n^{\infty} \frac{4}{x^5} dx} = \lim_{t \to \infty} \int_n^t \frac{4}{x^5} dx =$$

$$= \lim_{t \to \infty} \left[- x^{-4} \right]_n^t = \lim_{t \to \infty} \left(-\frac{1}{t^4} + \frac{1}{n^4} \right) = \frac{1}{n^4}$$

We need $s - s_n \leq \frac{1}{10^8}$.
The inequality holds if $n^4 \geq 10^8$

$n \geq 100$.

$n = 100$ is the minimal number, for which we know the inequality (by the integral test).

6. (32 points) Determine whether the following series converge absolutely, converge conditionally, or diverge. Name each test you use and indicate why all the conditions needed for it to apply actually hold.

8 pts

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n+10}}{n^2+3n+5}$ $= a_n$

Use limit comparison test with $\sum b_n$, $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

The latter is convergent, by the p-test: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\sqrt{n+10}\right)/\sqrt{n}}{\left(n^2+3n+5\right)/n^2} = \lim_{n\to\infty} \left(\frac{\sqrt{1+\frac{10}{n}}}{1+\frac{3}{n}+\frac{5}{n^2}} \right) = 1 < \infty$$

Hence, $\sum a_n$ converges, as well. (absolutely)

8 pt

(b) $\sum_{n=1}^{\infty} \underbrace{(-1)^n e^{1/n}}_{a_n}$

$$\lim_{n\to\infty} \left| (-1)^n e^{\frac{1}{n}} \right| = e^{\left(\lim_{n\to\infty} \frac{1}{n}\right)} = e^0 = 1 \neq 0$$

The series diverges, by the n-term test.

8 pts

(c) $\sum_{n=3}^{\infty}(-1)^n\left(\frac{\ln(n)}{n}\right)$

$\frac{\ln(n)}{n} \geq \frac{1}{n}$ for $n \geq 3$. $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (harmonic series)

Hence $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ diverges. The series is NOT absolutely convergent.

Let us check the conditions of the Alternating Series Theorem:

(1) $\lim_{n\to\infty} \frac{\ln(n)}{n} \underset{\text{L'Hop}}{\overset{[\frac{\infty}{\infty}]}{=}} \lim_{n\to\infty} \frac{(1/n)}{1} = 0$ ✓

(2) Sign is alternating ✓

(3) Decreasing! $\frac{\partial}{\partial x}\left(\frac{\ln(x)}{x}\right) = \frac{(1/x)\cdot x - \ln(x)}{x^2} < 0$ for $x \geq 3$. ✓

Hence, the series is CONDITIONALLY CONVER.

(d) $\sum_{n=1}^{\infty}(-1)^n \underbrace{\left(\frac{n!2^n}{(2n)!}\right)}_{a_n}$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{(n+1)!\ 2^{n+1}}{(2n+2)!}\right]\cdot\left[\frac{(2n)!}{n!\ 2^n}\right] =$$

$$= \lim_{n\to\infty} \frac{(n+1)\cdot 2}{(2n+2)(2n+1)} = 2\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}+\frac{1}{n}\right)}{\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)} =$$

$$= 0 < 1,$$

Hence, the series is $\boxed{\text{absolutely convergent}}$, by the Ratio-Test.