Three descriptions of the projection from a linear subspace of $\mathbb{P}^n$

Let V be an $n+1$ dimensional vector space over a field k and E a subspace. Here are three ways to describe the projection morphism

$$\pi : \mathbb{P}(V) \setminus \mathbb{P}(E) \to \mathbb{P}(V/E).$$

Each has its advantage.

1 Coordinate free description

A point $q \in \mathbb{P}(V) \setminus \mathbb{P}(E)$ determines a 1-dimensional subspace Q of V, which is not contained in E, so $(Q+E)/E$ is a one-dimensional subspace of V/E, hence a point $\pi(q)$ of $\mathbb{P}(V/E)$. This desciption is the most natural, it does not depend on any choices. $\mathbb{P}(V/E)$ parameterizes subspaces of V containing E of dimension one larger than E.

2 Geometric description

Choose a subspace W of V of complementary dimension, such that $E \cap W = (0)$. Then the quotient linear transformation $V \to V/E$ restricts to W as an isomorphism onto V/E and so induces an isomorphism $\mathbb{P}(W) \cong \mathbb{P}(V/E)$. Thus, the projection is a morphism

$$\pi : \mathbb{P}(V) \setminus \mathbb{P}(E) \to \mathbb{P}(W).$$

A point $q \in \mathbb{P}(V) \setminus \mathbb{P}(E)$ determines a 1-dimensional subspace Q of V and $\pi(q)$ is the unique point of intersection $\mathbb{P}(Q+E) \cap \mathbb{P}(W)$.

3 Coordinate dependent description

Choose a basis for V, so homogeneous coordinates $x_0, \ldots, x_n$ for $\mathbb{P}(V) \cong \mathbb{P}^n$. Conceptually, $\{x_0, \ldots, x_n\}$ are the dual basis of V^*. Choose a basis $\{L_0, \ldots, L_s\}$ of the subspace $(V/E)^*$ of V^*. We get coordinates on $\mathbb{P}(V/E)$, so an isomorphism $\mathbb{P}(V/E) \cong \mathbb{P}^s$, and the projection is a morphism

$$\pi : \mathbb{P}(V) \setminus \mathbb{P}(E) \to \mathbb{P}^s.$$

Now each $L_j(x_0, \ldots, x_n)$ is a linear combination of the x_i's so a homogeneous polynomial of degree 1, and $\mathbb{P}(E) = V(L_0, \ldots, L_s)$. The morphism π is then given by

$$\pi(q) = (L_0(q) : L_1(q) : \cdots : L_s(q)).$$

This way we see that π is indeed a morphism. A limitation of this description is that if $X \subset \mathbb{P}(V)$ is a subvariety not contained in $\mathbb{P}(E)$, then the restriction of π to X is a rational map $\varphi : X \to \mathbb{P}(V/E)$ and its domain of definition may include some points of $X \cap \mathbb{P}(E)$. This is the case, for example, if $\mathbb{P}(E)$ is a point p which is a smooth point of a curve X, so that $\varphi(p)$ is the tangent line L to X at p (in the coordinate free description) or the intersection point of L with the hyperplane $\mathbb{P}(W)$ (in the geometric description).