Algebraic Geometry Math 797 Homework Assignment 9, Fall 2021

The field k below is assumed algebraically closed.

1. (Shafarevich, Ch III, Sec. 2 Problem 2) Let D be an effective divisor of positive degree on a non-singular non-rational projective curve X. Show that $\dim_k(\mathcal{L}(D)) \leq \deg(D)$. This improves the bound in Theorem 6 in Lecture 25.

2. (Shafarevich, Ch III, Sec. 2 Problems 3, 4, 5, 6, 7, 8, 9) Let X be the projective closure $V(\tilde{z}\tilde{y}^2 - \tilde{x}^3 - A\tilde{x}\tilde{z}^2 - B\tilde{z}^3) \subset \mathbb{P}^2$ of the affine plane curve $y^2 = x^3 + Ax + B$, where the polynomial $x^3 + Ax + B$ does not have multiple roots and $char(k) \neq 2$.

   (a) Show that X is non-singular and that the intersection of X with the line at infinity consists of the single point $p = (0 : 1 : 0)$. Denote the restrictions to X of the affine coordinate functions x and y by x, y as well. Show that $\frac{x}{y}$ is a local parameter at p and that $\nu_p(x) = -2$ and $\nu_p(y) = -3$.

   (b) Every function f in $K(X)$ is of the form $P(x) + Q(x)y$. Determine when is f in $\mathcal{L}(mp)$ and show that $\ell(mp) = m$, for $m \geq 0$.

   (c) Let $p_1, p_2 \in X$ and denote by $C_{p_i} \in Cl(X)$ the divisor classes of $p_i - p$, $i = 1, 2$. There exists a point $p_3 \in X$, such that $C_{p_3} = C_{p_1} + C_{p_2}$, by Theorem 5 in Lecture 25. Find the coordinates of p_3 in terms of the coordinates of p_1 and p_2.

   (d) Given three points p_1, p_2, p_3 in X, show that $C_{p_1} + C_{p_2} + C_{p_3} = 0$, if and only if the three points are collinear.

   (e) Show that $-C_{(a:b:1)} = C_{(a:-b:1)}$, for $(a : b : 1) \in X$. Show that $Cl^0(X)$ has exactly four elements of order 2.

   (f) Given a point $q \in X$, denote by $\Theta_q X$ the line in $\mathbb{P}^2$ tangent to X at q. The point q is an *inflection point*, if the multiplicity $m_q(X, \Theta_q X)$ of q as an intersection point is ≥ 3. Show that q is an inflection point, if and only if $3C_q = 0$.

   (g) Show that the line passing through two inflection points of X intersects it in a third inflection point.

3. (Shafarevich, Ch III, Sec. 4 Problems 1 and 2) Assume that $char(k) \neq 2$. Let X be the affine curve given by $x^2 + y^2 = 1$. Denote the restrictions of the affine coordinates to X by x and y as well.

   (a) Show that the differential form dx/y is regular on X.

   (b) Show that $\Omega^1[X] = \Gamma(X)dx/y$.

   Hint: Use the fact that $dx/y + dy/x = \left(\frac{1}{2xy}\right) d(x^2 + y^2) = 0$.

4. (Shafarevich, Ch III, Sec. 4 Problem 4) Show that $\Omega^n(\mathbb{P}^n) = 0$. Hint: Use Theorem 3 in Lecture 26.

5. (Shafarevich, Ch III, Sec. 4 Problem 5) Show that $\Omega^1(\mathbb{P}^n) = 0$.

6. (Shafarevich, Ch III, Sec. 4 Problem 6) Let $(x_0 : x_1)$ be the homogeneous coordinates on $\mathbb{P}^1$ and set $t := x_1/x_0$. Let $P(t) = \prod_i (t - a_i)^{d_i}$ and $Q(t) = \prod_j (t - b_j)^{e_j}$ be relatively prime polynomials with $\deg(P) = m$ and $\deg(Q) = n$. At what points is the form $\omega := \frac{P(t)}{Q(t)} dt$ not regular? Find the divisor $(\omega) \in \mathrm{Div}(\mathbb{P}^1)$ of ω.

7. (Shafarevich, Ch III, Sec. 5 Problems 2 (modified), 3, 4, 5) Suppose that $char(k) = 0$.

   (a) Let $\varphi : X \to Y$ be a surjective morphism of non-singular projective curves. Let p be a point of X, set $q := \varphi(p)$, and let t be a local parameter of Y at q. Show that the integer $e_p := \nu_p(\varphi^* dt)$ does not depend on the choice of t and that $e_p > 0$, if and only if the multiplicity of p in the divisor $\varphi^*(q) \in \mathrm{Div}(X)$ is greater than 1. The integer e_p is called the *ramification index* of p and p is called a *ramification point* if $e_p > 0$.

   (b) Let $\varphi^*(q) = \sum_i l_i p_i \in \mathrm{Div}(X)$, $l_i \in \mathbb{Z}$, $p_i \in X$. Show that $e_{p_i} = l_i - 1$.

   (c) Suppose that $Y = \mathbb{P}^1$. Show that $g(X) = \left(\frac{1}{2}\sum_{p \in X} e_p\right) - \deg(\varphi) + 1$. Hint: Use Theorem 1 of Lecture 24 and the fact that $\deg(K_X) = 2g(X) - 2$.

   (d) For Y of arbitrary genus, show that
$$[2g(X) - 2] = \left(\sum_{p \in X} e_p\right) + \deg(\varphi)[2g(Y) - 2].$$

   (e) Let ω be a rational differential 1-form on Y. Show that if $\varphi^*(\omega)$ is regular, the ω is regular.

8. (Shafarevich, Ch III, Sec. 5 Problem 9) Verify the Riemann-Roch theorem for $X = \mathbb{P}^1$.

9. (Shafarevich, Ch III, Sec. 5 Problems 10, 11) Let X be a smooth projective curve of genus 1 and p a point of X.

   (a) Show that for every $n > 1$ there exists a rational function $u_n \in K(X)$, which is regular on $X \setminus \{p\}$, and for which $\nu_p(u_n) = -n$. Hint: Use the fact that $\ell(K_X - D) = 0$, if $\deg(D) > \deg(K_X) = 2g(X) - 2$.

   (b) Show that the subfield $k(u_2, u_3)$ of $K(X)$ is equal to $K(X)$ and that there exists a polynomial $F(x, y)$ of degree 3, such that $F(u_2, u_3) = 0$. Hint: Apply Riemann-Roch to $\mathcal{L}(6p)$. Prove that $[K(X) : k(u_2, u_3)] = 1$ using Theorem 1 of Lecture 24.

   (c) Assume that $char(k) \neq 2, 3$. Show that every curve of genus 1 is isomorphic to a curve given by the equation
$$y^2 = x^3 + Ax + B,$$
   where $\mathcal{L}(2p) = \mathrm{span}_k\{1, x\}$ and $\mathcal{L}(3p) = \mathrm{span}_k\{1, x, y\}$.