Algebraic Geometry Math 797 Homework Assignment 8, Fall 2021

The field k below is assumed algebraically closed.

1. (Shafarevich, Ch III, Sec. 1 Problem 3) Let X be the affine plane curve $V(x^2+y^2-1)$, $f : X \to \mathbb{A}^1$ the projection $f(x,y) = x$, and D_a the prime divisor corresponding to the point in $\mathbb{A}^1$ with coordinate a. Determine the inverse image divisor $f^*(D_a)$ in $\mathrm{Div}(X)$.

2. (Shafarevich, Ch III, Sec. 1 Problem 4) Let X be a smooth projective curve and $f \in K(X) \setminus \{0\}$ a rational function. Regarding f as a morphism $f : X \to \mathbb{P}^1$, prove that $(f) = f^*(D)$, where D is the divisor $0 - \infty$ on $\mathbb{P}^1$.

3. (Shafarevich, Ch III, Sec. 1 Problem 5) Let X be a non-singular affine variety. Show that $Cl(X) = 0$, if and only if the ring $\Gamma(X)$ of regular functions is a unique factorization domain. Hint: Theorem 47 page 141 in Matsumura's *Commutative Algebra* states that a noetherian integral domain A is a unique factorization domain, if and only if every prime ideal of height[1] 1 in A is principal. You will also need to use the Lemma on page 5 of the Lecture 22 notes.

4. (Shafarevich, Ch III, Sec. 1 Problem 7) Compute $Cl(\mathbb{P}^n \times \mathbb{A}^n)$ for $n > 0$.

5. (Shafarevich, Ch III, Sec. 1 Problem 11) Let X be the projective plane curve $V(y^2z - x^2(z+x))$. Note that X has a node at $q := (0:0:1)$ and the strict transform of q via the blow-up $\varphi : \mathbb{P}^1 \to X$ centered at q consists of two points q_1 and q_2 in $\mathbb{P}^1$. Show that every Cartier divisor on X is linearly equivalent to a Cartier divisor whose support does not contain the point q. Use this to describe $\mathrm{Pic}(X)$ as G/P, where $G = \mathrm{Div}(\mathbb{P}^1 \setminus \{q_1, q_2\})$ and P is the group of those principal divisors (f) for which f is regular at q_1 and q_2 and $f(q_1) = f(q_2) \neq 0$. Show that $\mathrm{Pic}^0(X)$ is isomorphic to the multiplicative group $k \setminus \{0\}$. Here $\mathrm{Pic}^0(X)$ is the kernel of the pullback from $\mathrm{Pic}(X)$ to its blow-up $\mathbb{P}^1$.

6. (Shafarevich, Ch III, Sec. 1 Problem 13) Let X be the quadric cone $V(z^2 - xy) \subset \mathbb{A}^3$ and $\varphi : \mathbb{A}^2 \to X$ be the morphism given by $\varphi(u,v) = (u^2, v^2, uv)$. We have the isomorphisms $\mathrm{Div}(\mathbb{A}^2) \cong \mathrm{Div}(\mathbb{A}^2 \setminus \{(0,0)\})$ and $\mathrm{Div}(X) \cong \mathrm{Div}(X \setminus \{(0,0,0)\})$ and the pullback $\varphi^* : \mathrm{Div}(X \setminus \{(0,0,0)\}) \to \mathrm{Div}(\mathbb{A}^2 \setminus \{(0,0)\})$ is well defined, since the target is non-singular and φ is dominant. Hence, we get the well defined pullback homomorphism $\varphi^* : \mathrm{Div}(X) \to \mathrm{Div}(\mathbb{A}^2)$.

   (a) Show that a divisor $D = (F) \in \mathrm{Div}(\mathbb{A}^2)$ belongs to the image $\varphi^*(\mathrm{Div}(X))$, if and only if $F(-u,-v) = \pm F(u,v)$, i.e., F is either an even or an odd function.

   (b) Let $P(X) \subset \mathrm{Div}(X)$ be the subgroup of principal divisors. Show that $\varphi^*(P(X))$ consists of principal divisors of even functions.

   (c) Show that $Cl(X)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

[1] The height of a prime ideal P is the maximal length h of a sequence $P_0 \subset P_1 \subset \cdots \subset P_h = P$ of prime ideals and strict inclusions. Note that $P_0 = (0)$ if A is a domain.

7. (Shafarevich, Ch III, Sec. 1 Problem 16) Show that every automorphism of $\mathbb{P}^n$ carries divisors of hyperplanes (prime divisors of degree 1) into each other. Hint: The class of hyperplanes in $Cl(\mathbb{P}^n)$ is the effective generating class.

8. (Shafarevich, Ch III, Sec. 1 Problem 17) Show that every automorphism of $\mathbb{P}^n$ is a linear projective transformation, i.e., an element of $PGL(n+1)$. Hint: Use problem 7.

9. (a) Let X be a variety such that its singular locus X_{sing} has codimension ≥ 2 in X. Let Z be a closed subset and set $U := X \setminus Z$.
   i. There is a surjective homomorphism $\rho : Cl(X) \to Cl(U)$ defined by
   $$\sum n_i Y_i \mapsto \sum n_i (Y_i \cap U),$$
   where we omit those $Y_i \cap U$ which are empty.
   ii. If the co-dimension of Z in X is ≥ 2, then the homomorphism ρ is an isomorphism.
   iii. If Z is an irreducible subset of co-dimension 1, then there is a right exact sequence
   $$\mathbb{Z} \to Cl(X) \to Cl(U) \to 0,$$
   where the first homomorphism maps 1 to $1 \cdot Z$.

   (b) If Y is an irreducible hypersurface of degree d in $\mathbb{P}^n$, $n \geq 2$, then $Cl(\mathbb{P}^n \setminus Y) = \mathbb{Z}/d\mathbb{Z}$.

10. (Shafarevich, Ch III, Sec. 1 Problem 18) Let Y be a non-singular variety of dimension $n \geq 2$ and $\sigma : X \to Y$ the blow-up centered at a point $y \in Y$.

   (a) Let $\iota : E \to X$ be the inclusion of the exceptional divisor. Recall that $E \cong \mathbb{P}(T_y Y) \cong \mathbb{P}^{n-1}$. Denote by $[E] \in Cl(X)$ the class of the prime divisor E. Show that $\iota^*[E]$ is the generator of degree -1 of $Cl(E) \cong Cl(\mathbb{P}^{n-1})$. Hint: Let $Z \subset Y$ be an irreducible subvariety of codimention 1, such that y is a non-singular point of Z. Denote by $[Z] \in Cl(Y)$ the class of Z and by $[\widetilde{Z}] \in Cl(X)$ the class of the strict transport of Z. Show that $\sigma^*([Z]) = [E] + [\widetilde{Z}]$, $\iota^*[\tilde{Z}]$ is a generator of $Cl(E)$, and that $\iota^*[E] = -\iota^*[\widetilde{Z}]$. Theorem 1 of Lecture 23 would be useful.

   (b) Show that $Cl(X) = Cl(Y) \oplus \mathbb{Z}$.