Algebraic Geometry Math 797 Homework Assignment 7, Fall 2021

The field k below is assumed algebraically closed.

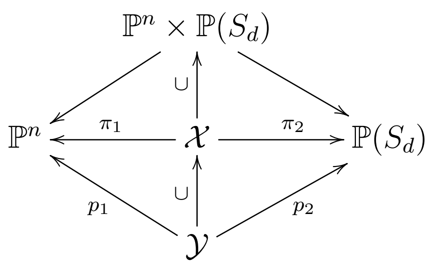
1. (Shafarevich Ch. II Sec. 1 Q2) Let X be the affine plane curve $V(y^2 - x^3)$. The morphism $\varphi : \mathbb{A}^1 \to X$, given by $\varphi(t) = (t^2, t^3)$, determines a birational isomorphism of the curve X and $\mathbb{A}^1$. What rational functions of t correspond to functions of the local ring $\mathcal{O}_p$ of the point $p = (0, 0)$ of X?

2. (a) (Shafarevich Ch. II Sec. 1 Q9) Show that a hypersurface of degree[1] 2 having a singular point p is a cone (a union of lines passing through p). Hint: Use Equation (1).

   (b) (Shafarevich Ch. II Sec. 1 Q10) Show that if a hypersurface of degree 3 has two singular points, then the line containing them lies on the hypersurface.

   (c) (Shafarevich Ch. II Sec. 1 Q11) Show that if a plane curve of degree 3 has 3 singular points, then it is the union of three lines.

3. (Shafarevich Ch. II Sec. 1 Q17) Assume that the ground field k has characteristic 0. Let d and n be integers satisfying $d \geq 2$ and $n \geq 2$. Let S_d be the graded summand of $k[x_0, \ldots, x_n]$ of degree d. Show that points in $\mathbb{P}(S_d)$ corresponding to hypersurfaces in $\mathbb{P}^n$ having a singular point form an irreducible hypersurface in $\mathbb{P}(S_d)$. Hint: Use the discussion about the universal hypersurface $\mathcal{X}$ in Lecture 17. Let $\mathcal{Y} \subset \mathcal{X}$ be the subset of points (x, F), such that x is a singular point of $V(F) \subset \mathbb{P}^n$. First show that $\mathcal{Y}$ is a Zariski closed subset. Next show that $\mathcal{Y}$ is irreducible by showing that the fibers of p_1 are all projective subspaces of $\mathbb{P}(S_d)$ of the same dimension.



4. (Shafarevich Ch. II Sec. 3 Q1) Let X be a curve, $p \in X$ a non-singular point, and $m_p \subset \mathcal{O}_p$ the maximal ideal in the local ring at p. Let $t \in m_p$ be an element which image in m_p/m_p^2 spans the latter. So $m_p = \{ct + g \ : \ c \in k, \ g \in m_p^2\}$. Show that every function $f \in \mathcal{O}_p$ can be represented uniquely in the form $f = ut^n$,

[1] A hypersurface in $\mathbb{A}^n$ or $\mathbb{P}^n$ is a Zariski closed subset of the form $X = V(F)$ and the degree of X is that of F. If X is a hypersurface in $\mathbb{P}^n$ and L is a line in $\mathbb{P}^n$ which is not contained in X, then

$$\deg(X) = \sum_{p \in L \cap X} m_p(L, X), \tag{1}$$

where $m_p(X, L)$ is the multiplicity of p as an intersection point, by the definition of $m_p(L, X)$ given in Lecture 18.

where $n \geq 0$ and u is an invertible element of $\mathcal{O}_p$. Deduce that $\mathcal{O}_p$ is a unique factorization domain. Hint: Use Nakayama's Lemma.

5. (Shafarevich Ch. II Sec. 3 Q5) Let X be the cone $V(x^2+y^2-z^2) \subset \mathbb{A}^3$. Show that the line $L = V(x, y-z)$ does not have a single local equation in any neighborhood of the point $(0,0,0)$. Hint: Use Nakayama's Lemma.

6. (Hartshorne Exercise I.5.1) Locate the singular points of the following curves in $\mathbb{A}^2$ (assume that the characteristic of k is not equal to 2). a) $x^2 = x^4 + y^4$, b) $xy = x^6 + y^6$, c) $x^3 = y^2 + x^4 + y^4$, d) $x^2y + xy^2 = x^4 + y^4$. Sketch these curves when $k = \mathbb{R}$. The figures from Hartshorne are included below.

7. (Hartshorne Exercise I.5.2) Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^2$. a) $xy^2 = z^2$, b) $x^2 + y^2 = z^2$, c) $xy + x^3 + y^3 = 0$. The figures from Hartshorne are included below.

8. (Hartshorne Exercise I.5.3) Let $Y \subset \mathbb{A}^2$ be a curve defined by the equation $f(x,y) = 0$. Let $P = (a,b)$ be a point of $\mathbb{A}^2$. Make a linear change of coordinates so that P becomes the point $(0,0)$. Then write f as a sum $f = f_0 + f_1 + \cdots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. The *multiplicity* of P on Y, denoted $\mu_P(Y)$, is the least r, such that $f_r \neq 0$. (Note that $P \in Y \Leftrightarrow \mu_p(Y) > 0$.) The linear factors of f_r are called the *tangent directions at* P.

   (a) Show that $\mu_P(Y) = 1 \Leftrightarrow P$ is a non-singular point of Y.

   (b) Find the multiplicity of each of the singular points in Question 6 above.

9. (Hartshorne, Problem I.5.6) *Blowing up curve singularities*). Let $Y = V(f)$ be an affine plane curve and $P = (a,b)$ a point of $\mathbb{A}^2$. Write $f = f_\mu + f_{\mu+1} + \ldots + f_d$, where f_i is a homogeneous polynomial of degree i in $(x-a)$ and $(y-b)$, and $f_\mu \neq 0$. Recall that the multiplicity of P on Y is μ. If $\mu > 0$, the tangent directions are cut out by the linear factors of f_μ. A *double point* is a point of multiplicity 2. We define a *node* (also called an *ordinary double point*) to be a double point with distinct tangent directions. Denote by $\varphi : \widetilde{Y} \to Y$ the morphism of blowing-up $P \in Y$.

   (a) Let Y be the cuspidal curve $V(y^2 - x^3)$ or the nodal curve $V(x^6 + y^6 - xy)$ from Question 6. Show that the curve $\widetilde{Y}$, obtained by blowing up Y at the point $O := (0,0)$, is non-singular. Note: The term *cusp* is defined in Exercise I.5.14 part d in Hartshorne. It is characterised also as a double point planar singularity $p \in Y$, such that $\varphi^{-1}(P)$ consists of a single point $\widetilde{P} \in \widetilde{Y}$ and $\widetilde{Y}$ is non-singular at $\widetilde{P}$.

   (b) Let P be a node on a plane curve Y. Show that $\varphi^{-1}(P)$ consists of two distinct non-singular points on $\widetilde{Y}$. We say that "blowing-up P resolves the singularity at P".

(c) Let $P = (0,0)$ be the tacnode of $Y = V(x^4 + y^4 - x^2)$ from Question 6. Show that $\varphi^{-1}(P)$ is a node. Using 9b we see that the tacnode can be resolved by two succesive blowing-up.

(d) Let Y be the plane curve $V(y^3 - x^5)$, which has a higher order cusp at O. Show that O is a triple point; that blowing-up O gives rise to a double point, and that one further blowing-up resolves the singularity.

10. (Hartshorne, Problem I.5.7) Let $Y \subset \mathbb{P}^2$ be a non-singular plane curve of degree > 1, defined by the equation $f(x,y,z) = 0$. Let $X \subset \mathbb{A}^3$ be the affine variety defined by f (this is the cone over Y). Let $P = (0,0,0)$ be the vertex of the cone and $\varphi : \widetilde{X} \to X$ the blowing-up of X at P.

    (a) Show that P is the only singular point of X.

    (b) Show that $\widetilde{X}$ is non-singular (cover it with open affine subsets).

    (c) Show that $\varphi^{-1}(P)$ is isomorphic to Y.

11. (Shafarevich Ch. II Sec. 5 Q4) Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^4$ be the rational map given by the formula $\varphi(x_0 : x_1 : x_2) = (x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2)$. Show that φ is a birational isomorphism and its inverse $\overline{\varphi(\mathbb{P}^2)} \to \mathbb{P}^2$ is the blow-up at $(1:0:0)$.