Algebraic Geometry Math 797 Homework Assignment 6, Fall 2021

The field k below is assumed algebraically closed.

1. (Shafarevich, Ch. I Sec. 6 problems 6, 7, 8) Let $X \subset \mathbb{A}^3$ be an algebraic curve (an irreducible affine algebraic variety of dimension 1). In parts 1a and 1b assume that X is not a line. Let x, y, z be the coordinates on $\mathbb{A}^3$.

   (a) Show that there exists a non-zero polynomial $f(x, y)$ that vanishes at all points of X. Show that all such polynomials form a principal ideal $(g(x, y))$ and that the curve $V(g)$ is the closure of the projection of X to the (x, y)-plane parallel to the z-axis. Provide a careful justification citing the precise theorems needed to show that the closure of the projection of X is a one dimensional variety.

   (b) Let $h(x, y, z) = g_0(x, y)z^n + \cdots + g_n(x, y) \in k[x, y][z]$ be a polynomial of smallest possible degree in z in the ideal $I(X)$, which is not divisible by g. Show that every $f \in I(X)$ of degree m in z can be written in the form

   $$f \cdot g_0^m = h \cdot Q + v,$$

   with $v \in k[x, y, z]$ divisible by g (and $\deg_z(v) < \deg_z(h)$). Deduce that $V(g, h)$ is an algebraic subset of dimension 1 which is itself contained in the union of X and finitely many lines parallel to the z-axis given by $V(g, g_0)$,

   $$X \subset V(g, h) \subset X \cup V(g, g_0).$$

   Now show that every irreducible component of $V(g, h)$ is 1-dimensional, and so $V(g, h) = X \cup L_1 \cup \cdots \cup L_r$, where the lines $\{L_1, \ldots, L_r\}$ are a subset of those in $V(g, g_0)$.

   (c) Show that every curve X in $\mathbb{A}^3$ can be determined by three equations. Hint: Let Y_j be the union of X and all the L_i except L_j. Note that $Y_j \cap L_j = X \cap L_j$. Use the fact that the ring $\Gamma(L_j)$ of regular functions on L_j is a principal ideal domain ($\cong k[t]$) to show that there exists an element $q_j \in I(Y_j) \subset k[x, y, z]$, such that $V(q_j) \cap L_j = X \cap L_j$. Next show that $q := \sum_{j=1}^{r} q_j$ satisfies $V(g, h, q) = X$.

2. Let $\mathfrak{gl}(n, k)$ be the variety of $n \times n$ matrices with entries in the field k, and let $char : \mathfrak{gl}(n, k) \to \mathbb{A}^n$ be the morphism, which takes a matrix A to the coefficients $(a_1, \ldots, a_n)$ of its characteristic polynomial $\det(A - xI) = (-1)^n x^n + a_1 x^{n-1} + \cdots + a_n$. Recall that the morphism *char* is surjective, since the companion matrix of a monic polynomial $f(x)$ has characteristic polynomial $f(x)$.

   (a) Given $A \in \mathfrak{gl}(n, k)$, denote by $\varphi_A : GL(n, k) \to \mathfrak{gl}(n, k)$ the morphism $\varphi_A(g) = gAg^{-1}$. (The formula for g^{-1} via the adjoint matrix shows that φ_A is indeed a morphism of affine varieties). Show that the dimension of the closure $\overline{Im(\varphi_A)}$ is $n^2 - \dim(C(A))$, where $C(A) := \{g \ : \ gAg^{-1} = A\}$ is the centralizer of A.

(b) Assume that A is nilpotent. Show that $\dim(C(A)) \geq n$, and that equality holds if and only if $\text{rank}(A) = n - 1$. Hint: Consider the Jordan canonical form of A.

(c) Show that the fiber $char^{-1}(0)$ is irreducible of dimension $n^2 - n$. Hint: Consider Theorem 6 in Lecture 17.

(d) Prove that all the fibers of $char$ are of pure[1] dimension $n^2 - n$. Hint: Use the k^*-equivariance of $char$ with respect to the standard k^*-action on $\mathfrak{gl}(n, k)$, the k^*-action $\lambda(a_1, \ldots, a_n) = (\lambda a_1, \ldots, \lambda^n a_n)$ on $\mathbb{A}^n$, and the Upper-Semi-Continuity Theorem for fiber dimension (Lecture 17) to reduce the question to the nilpotent case.

Remark: It is not hard to show that in fact all fibers of $char$ are irreducible. Indeed, the Jordan canonical form theorem shows first that each fiber is the union of finitely many $GL(n, k)$ orbits, each of which is irreducible, and precisely one of them is $(n^2 - n)$-dimensional.

3. Construction of the Grasmannian variety $G(r, n)$: Let V be an n-dimensional vector space over k and $\overset{r}{\wedge} V$ its exterior product. Recall that $\dim\left(\overset{r}{\wedge} V\right) = \binom{n}{r}$. If $\{e_1, \ldots, e_n\}$ is a basis for V, then $\overset{r}{\wedge} V$ has the basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} \ : \ \text{where } i_1 < \cdots < i_r \text{ and } 1 \leq i_j \leq n\}. \tag{1}$$

Let $G(r, n)$ be the set of r dimensional subspaces of V. Consider the set theoretic map

$$[\bullet] \ : \ G(r, n) \quad \longrightarrow \quad \mathbb{P}\left(\overset{r}{\wedge} V\right) \cong \mathbb{P}^{\binom{n}{r} - 1}$$

sending an r-dimensional subspace W of V to the point $[W] \in \mathbb{P}\left(\overset{r}{\wedge} V\right)$, corresponding to the line $\overset{r}{\wedge} W$ in $\overset{r}{\wedge} V$. The basis (1) introduces homogeneous coordinates on $\mathbb{P}\left(\overset{r}{\wedge} V\right)$, called *Plücker coordinates*. The Plücker coordinates of $[W]$ can be computed in terms of a basis $\{f_1, \ldots, f_r\}$ of W as the coefficients on the right hand side of the following equation

$$f_1 \wedge \cdots \wedge f_r \quad = \sum_{1 \leq i_1 < \cdots < i_r \leq n} p[i_1, \ldots, i_r] e_{i_1} \wedge \cdots \wedge e_{i_r}.$$

Note that $p[i_1, \ldots, i_r]$ is an $r \times r$ minor of the matrix, whose columns are the coordinate vectors of $f_1, \ldots, f_r$ in the chosen basis for V. A non-zero vector in $\overset{r}{\wedge} V$ is called *decomposeable*, if it is of the form $f_1 \wedge \cdots \wedge f_r$, for some r independent vectors in V. Denote by $D(r, n) \subset \mathbb{P}\left(\overset{r}{\wedge} V\right)$ the subset of all lines spanned by decomposable vectors. Clearly $D(r, n)$ is equal to the image of $[\bullet]$.

[1] An algebraic set has pure dimension d if all its irreducible components are of dimension d.

(a) Let $t \in \overset{r}{\wedge} V$ be a non-zero vector and $\varphi_t : V \to \overset{r+1}{\wedge} V$ the linear homomorphism sending $x \in V$ to $t \wedge x$. Prove that t is decomposable, if and only if $\dim \ker(\varphi_t) \geq r$. Hint: If $\dim \ker(\varphi_t) \geq r$, we may choose the basis for V so that $e_i \in \ker(\varphi_t)$, for $1 \leq i \leq r$.

(b) Prove that the map $[\bullet] : G(r,n) \to D(r,n)$ is bijective. We identify the two sets from now on and denote both by $G(r,n)$.

(c) Prove that $G(r,n)$ is a Zariski closed subset of $\mathbb{P}\left(\overset{r}{\wedge} V\right)$. Hint: Use part 3a.

(d) Let $L_0 := \text{span}\{e_1, \dots, e_r\}$ and consider the map $q : GL(n,k) \to G(r,n)$ given by $T \mapsto T(L_0)$. Show that q is a surjective map and a morphism. Hint: Explicitly describe the Plücker coordinates of $q(T)$ in terms of the first r columns of the invertible matrix T.

(e) Prove that $G(r,n)$ is an irreducible projective variety of dimension $r(n-r)$.

(f) Let $U_{[i_1,\dots,i_r]} \subset \mathbb{P}\left(\overset{r}{\wedge} V\right)$ be the open subset where the Plücker coordinate $p[i_1,\dots,i_r]$ does not vanish. Prove that $G(r,n) \cap U_{[i_1,\dots,i_r]}$ is isomorphic to $\mathbb{A}^{r(n-r)}$. Hint: Let $A \subset GL(n)$ be the subgroup consisting of matrices of the form $\begin{pmatrix} I_r & 0 \\ * & I_{n-r} \end{pmatrix}$, where I_r is the $r \times r$ identity matrix. Show that q restricts as an isomorphism from A onto $G(r,n) \cap U_{[1,\dots,r]}$. In order to show that the inverse of the restriction is a morphism you will need to show that if C is an $(n-r) \times r$ matrix then the (i,j) entry c_{ij} of C can be expressed, up to an explicit sign, as one of the $r \times r$ minors of the $n \times r$ matrix $\begin{pmatrix} I_r \\ C \end{pmatrix}$.

4. Let V be a $(2k+\epsilon)$-dimensional vector space, where $\epsilon = 0$ or 1, and $t \in \overset{2}{\wedge} V$. A standard fact from linear algebra states that there exists a basis $\{e_1, \dots, e_{2k+\epsilon}\}$ of V, with respect to which $t = \sum_{i=1}^{k} c_i e_{2i-1} \wedge e_{2i}$. Hence, anti-symmetric bilinear forms have *even* rank. Let V be a $2k$-dimensional vector space. The polynomial map $P : \overset{2}{\wedge} V \to \overset{2k}{\wedge} V$, given by $t \mapsto t^k$, is an element of $Sym^k(\overset{2}{\wedge} V)^* \otimes \overset{2k}{\wedge} V$. More explicitly, if we choose coordinates on V, then P is a polynomial of degree k in the coordinates of $\overset{2}{\wedge} V$, called the Pffafian.[2]

(a) Show that a vector $t \in \overset{2}{\wedge} V$ is decomposable, if and only if $t \wedge t = 0 \in \overset{4}{\wedge} V$.

(b) Prove that $G(2,4)$ is a quadric hypersurface in $\mathbb{P}^5$ and find its homogeneous quadratic equation in the Plücker coordinates.

[2]Given an element $t \in \overset{2}{\wedge} V$, denote by $T : V^* \to V$ the anti-self-dual linear transformation induced by t. Consider the map $\det : \overset{2}{\wedge} V \to (\overset{2k}{\wedge} V)^{\otimes 2}$ sending t to $\det(T) := \overset{2k}{\wedge} T$. Then det belongs to $Sym^{2k}(\overset{2}{\wedge} V)^* \otimes (\overset{2k}{\wedge} V)^{\otimes 2}$, i.e., det is a polynomial of degree $2k$ in the coordinates of $\overset{2}{\wedge} V$. It is easy to show, using the $GL(V)$-invariance of both, that the determinant is equal to a universal non-zero constant times the square of the Pffafian.

(c) Assume $\mathrm{char}(k) \neq 2$. Let $Q(x_0, \ldots, x_5)$ be a quadratic polynomial with a non-degenerate symmetric bilinear form. Prove that the quadric hypersurface $V(Q)$ in $\mathbb{P}^5$ is isomorphic to $G(2,4)$. Hint: See problem 7 in Homework 3.

5. (Based on Shafarevich, Ch I., Sec 6.4. Feel free to consult the text, though you should be able to work it out on your own.) Assume now that V is $n+1$ dimensional so that $\mathbb{P}V$ is isomorphic to $\mathbb{P}^n$. Choose homogeneous coordinates on $\mathbb{P}V$, let $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}V$, and let S_d be its graded summand of degree d. Set $\mathcal{H}(d,n) := \mathbb{P}S_d$. A point in $\mathcal{H}(d,n)$ parametrizes a hypersurface of degree d in $\mathbb{P}^n$. Let

$$I(r,n,d) \quad \subset \quad \mathcal{H}(d,n) \times G(r+1, n+1)$$

be the incidence subset, consisting of pairs (X, W), such that the r-dimensional linear subspace $\mathbb{P}W$ of $\mathbb{P}^n$ is contained in the hypersurface X. One easily checks that $I(r,n,d)$ is a Zariski closed subset of $\mathcal{H}(d,n) \times G(r+1,n+1)$.

(a) Show that the projection $p_2 : I(r,n,d) \to G(r+1,n+1)$ is surjective and its fiber over $W \in G(r+1,n+1)$ is a linear subspace of $\mathcal{H}(d,n)$ of dimension $\binom{n+d}{d} - \binom{r+d}{d} - 1$. Hint: Identify S_d with $\mathrm{Sym}^d V^*$ and consider the natural restriction homomorphism $\mathrm{Sym}^d V^* \to \mathrm{Sym}^d W^*$.

(b) Prove that $I(r,n,d)$ is an irreducible variety of dimension

$$(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1.$$

Hint: Consider a theorem in Lecture 17.

(c) Prove that the image of the first projection $p_1 : I(r,n,d) \to \mathcal{H}(d,n)$ is a closed subvariety of $\mathcal{H}(d,n)$. Hint: A one line argument!

(d) Assume that $(n-r)(r+1) < \binom{r+d}{d}$. Prove that $p_1(I(r,n,d))$ is a proper subset of $\mathcal{H}(d,n)$. Conclude that for $d \geq 4$, there is a dense open subset $\mathcal{H}'(d,3)$ in $\mathcal{H}(d,3)$, such that for $X \in \mathcal{H}'(d,3)$, the corresponding surface X of degree d in $\mathbb{P}^3$ does not contain any line.

(e) Show that every cubic surface in $\mathbb{P}^3$ contains a line. Hint: Set $n=3$, $r=1$, and $d=3$ and note that $\dim I(1,3,3) = \dim \mathcal{H}(3,3)$. Show first that the (singular) cubic $x_0x_1x_2 - x_3^3$ contains only 3 lines.

(f) Find 27 lines on the Fermat cubic surface $V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subset \mathbb{P}^3$.

Note: It can be proven that over the open subset of $\mathcal{H}(3,3)$, where X is smooth, the fiber $p_1^{-1}(X)$ consists of 27 points; representing 27 lines on X.