Algebraic Geometry Math 797 Homework Assignment 5, Fall 2021

The field k below is assumed algebraically closed. Recall that a variety is *affine* if it is isomorphic to a closed (necessarily irreducible) subset of $\mathbb{A}^n$, for some n. A variety is *projective* if it is isomorphic to a closed subset of $\mathbb{P}^n$, for some n.

1. (Shafarevich, Ch. I, Sec. 4 problem 7) Show that every rational map $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ is actually a morphism.

2. (Shafarevich, Ch. I, Sec. 5, Problem 5) Show that $\mathbb{P}^2 \setminus \{x\}$, x a point, is not isomorphic to an affine variety, nor to a projective variety. Hint: Compare with HW3 Q6.

3. (Shafarevich, Ch. I, Sec. 5, Problem 6) Show that $\mathbb{P}^1 \times \mathbb{A}^1$ is not isomorphic to an affine variety, nor to a projective variety.

4. (Shafarevich, Ch. I, Sec. 5, Problem 9) Let U and V be affine open subsets of a variety X. Show that the intersection $U \cap V$ is affine. Hint: Consider the intersection of $U \times V$ with the diagonal in $X \times X$.

5. (Shafarevich, Ch. I, Sec. 5, Problem 10) Let $S := k[x_0, \ldots, x_n]$ and S_d the subspace of homogeneous polynomials of degree d. Assume that $d = \ell m$, where ℓ and m are positive integers. Let $Z \subset \mathbb{P}(S_d)$ be the subset of points corresponding to lines in S_d which are spanned by ℓ powers. Show that Z is a closed subset.

6. (a) Let X and Y be quasi-projective varieties, let $\varphi : X \to Y$ be a dominant morphism, and let $h \in \Gamma(X)$ be a regular function on X with $V(h) \neq \emptyset$ and $V(h) \neq X$. Set $X_h := X \setminus V(h)$. Show that the restriction $\psi : X_h \to Y$ of φ to X_h is not a finite morphism. Hint: Reduce to the case where Y and X_h are affine and note that if $\Gamma(X_h)$ is integral over $\Gamma(Y)$, then it is integral over $\Gamma(X)$.

   (b) Let X be an affine variety, let p be a point of $\mathbb{A}^n$, $n \geq 2$, and let $\varphi : \mathbb{A}^n \setminus \{p\} \to X$ be a morphism. Show that φ is not a finite morphism. Hint: Show that φ extends to a morphism $\psi : \mathbb{A}^n \to X$ and then show that there does not exist an affine open neighborhood V of $\psi(p) \in X$, such that the open subset $\varphi^{-1}(V)$ of $\mathbb{A}^n \setminus \{p\}$ is affine. (Observe that the proof of HW3 Q6 goes through if we replace $\mathbb{A}^2$ in that question by any affine open subset U of $\mathbb{A}^n$, $n \geq 2$, and $(0, 0)$ by any point of U. You may use this observation).

   (c) Let X and Y be quasi-projective varieties and $\varphi : X \to Y$ a finite morphism. Let V be an open subset of Y and set $U := \varphi^{-1}(V)$. Show that the restriction of φ to U is a finite morphism onto V.

7. (Shafarevich, Ch. I, Sec. 5, Problem 8 modified)

(a) Let $F \in k[x,y]$ be an irreducible polynomial of degree $d > 1$ and set $X := V(F) \subset \mathbb{A}^2$. Given a homogeneous polynomial $H(x,y) \in k[x,y]$ of degree 1, let

$$\varphi_H : X \to \mathbb{A}^1 \tag{1}$$

be the restriction of H to X. Set $L := V(H) \subset \mathbb{A}^2$. Let $\overline{X}$ be the closure[1] of X in $\mathbb{P}^2$ with homogeneous coordinates $(x : y : z)$ and define $\overline{L}$ similarly. Let p be the unique point of intersection $p := V(z) \cap \overline{L}$. Show that if $p \notin \overline{X}$, then the morphism φ_H in (1) is finite. Hint: Show that the projection $\pi : \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1$ from the point p fits in a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi_H} & \mathbb{A}^1 \\ \cap\downarrow & & \downarrow\cap \\ \mathbb{P}^2 \setminus \{p\} & \xrightarrow[\pi]{} & \mathbb{P}^1 \end{array} \tag{2}$$

and use a theorem in Lecture 14.

(b) Keep the notation of part 7a. Let $G \in k[x,y,z]$ be $z^d F(x/z, y/z)$. Recall that $\overline{X} = V(G)$, by Homework 2 Q6(b). Assume that the point $p = V(z) \cap \overline{L}$ belongs to $\overline{X}$ and $\left(\frac{\partial G}{\partial x} : \frac{\partial G}{\partial y} : \frac{\partial G}{\partial z}\right)(p)$ is non-zero. Show that the morphism φ_H in (1) is not finite, if $\left(\frac{\partial G}{\partial x} : \frac{\partial G}{\partial y} : \frac{\partial G}{\partial z}\right)(p)$ is not equal[2] to $(0 : 0 : 1)$. Hint: (i) Show that p belongs to the domain of definition of the rational map $\varphi_H : \overline{X} \to \mathbb{A}^1$ (use the example on pages 9, 10 of Lecture 4 notes) and use problem 6a with $X \cup \{p\}$ as the domain of φ. (ii) You will need to construct a regular function h which vanishes at p to use problem 6a. This would be easier if you argue by contradiction and restrict the morphism to a smaller open subset containing p along the lines of problem 6c.

(c) Let $X = V(xy - 1) \subset \mathbb{A}^2$. Determine the set of homogeneous polynomial $H(x,y) \in k[x,y]$ of degree 1, for which the morphism φ_H in (1) is finite.

(d) Repeat part (7c) for $X = V(y^2 - x)$.

8. (Shafarevich, Ch I, Section 6, problem 1) Let L be an $(n-1)$-dimensional linear subspace of $\mathbb{P}^n$, let $X \subset L$ a closed (irreducible) subvariety, and let y be a point of $\mathbb{P}^n \setminus L$. Let $Y \subset \mathbb{P}^n$ be the union of all those lines in $\mathbb{P}^n$ passing through the point y and some point of X. Show that Y is an irreducible projective variety of dimension $\dim(X) + 1$.

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[1] Identify $\mathbb{A}^2$ with $\mathbb{P}^2 \setminus V(z)$ via $(x,y) \mapsto (x : y : 1)$.

[2] Equivalently, the tangent line $V\left(\frac{\partial G}{\partial x}(p)x + \frac{\partial G}{\partial y}(p)y + \frac{\partial G}{\partial z}(p)z\right)$ to $\overline{X}$ at p is not equal to $V(z)$. This means that the tangent line corresponds to a point of $\mathbb{A}^1$, since $V(z)$ corresponds to the point at infinity in $\mathbb{P}^1$ considering the inclusion of $\mathbb{A}^1$ in the right vertical arrow in diagram (2). You are proving that a necessary condition for φ_H to be finite is that X is not missing any point of $\overline{X}$ which belongs to the domain of definition of the rational map $\varphi_H : \overline{X} \to \mathbb{A}^1$. This condition is in fact sufficient, φ_H is finite if and only if $\left(\frac{\partial G}{\partial x} : \frac{\partial G}{\partial y} : \frac{\partial G}{\partial z}\right)(p) = (0 : 0 : 1)$, but the proof of the converse implication requires Theorem 11 in Ch II section 4 of Shafarevich.