Algebraic Geometry Math 797 Homework Assignment 4, Fall 2021

The field k below is assumed algebraically closed.

1. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties. Let $I(X) \subset k[x_1, \ldots, x_n]$ and $I(Y) \subset k[y_1, \ldots, y_m]$ be their ideals. Let $J \subset k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be the ideal generated by $I(X)$ and $I(Y)$. Recall that $X \times Y$ is the closed algebraic subset $V(J)$ of $\mathbb{A}^{n+m}$. Prove that $X \times Y$ is irreducible, hence an affine variety. Hint: Let $f, g \in k[x_1, \ldots, x_n, y_1, \ldots, y_m]$. Prove that if $f \notin I(X \times Y)$ and $g \notin I(X \times Y)$, then $fg \notin I(X \times Y)$ by proving the following statement. If there exists $(x_1, y_1) \in X \times Y$, such that $f(x_1, y_1) \neq 0$ and there exists $(x_2, y_2) \in X \times Y$, such that $g(x_2, y_2) \neq 0$, then there exists $(x_3, y_3) \in X \times Y$, such that $f(x_3, y_3) \neq 0$ and $g(x_3, y_3) \neq 0$. First find x_3, such that $f(x_3, y_1) \neq 0$ and $g(x_3, y_2) \neq 0$.

2. Let $F \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of positive degree. Prove that $\mathbb{P}^n_F := \{x \in \mathbb{P}^n \ : \ F(x) \neq 0\}$ is an affine variety. Hint: Consider the d-Uple embedding with $d = \deg(F)$. This is a straightforward generalization of Homework 3 Problem 10 and will not be graded.

3. (Hartshorne, Exercise I.3.7) Let X be a projective variety. We have proven that $\mathcal{O}(X) = k$ (all global regular functions on X are constant). Use this fact together with Problem 2 to solve the following:

   (a) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

   (b) Show that any two curves in $\mathbb{P}^2$ have a non-empty intersection.

   (c) More generally, show that if $X \subset \mathbb{P}^n$ is a projective variety of dimension[1] ≥ 1, and if $Y = V(F)$ is a hypersurface (where F is a homogeneous polynomial of positive degree), then $X \cap Y \neq \emptyset$.

4. Let X and Y be varieties and $\varphi : X \to Y$ a rational map. Recall that φ is an an equivalence class of pairs (U, φ_U), where U is a non-empty open subset of X and $\varphi : U \to Y$ is a morphism. Show that if $\varphi_U(U)$ is dense in Y and $(U, \varphi_U) \sim (V, \varphi_V)$, then $\varphi_V(V)$ is dense in Y. (In this case we called φ a *dominant* rational map).

5. (Fulton's *Algebraic curves* problem 4.28 modified) Set $A := k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$. A polynomial $F \in A$ is called *bi-homogeneous* of bi-degree (p, q), if F is homogeneous of degree p (resp. q) when considered as a polynomial in $X_0, \ldots, X_n$ (resp. in the Y_i's). Given a set S of bi-homogeneous polynomials, set

$$V(S) \ := \ \{(x, y) \in \mathbb{P}^n \times \mathbb{P}^m \ : \ F(x, y) = 0, \text{ for all } F \in S\}.$$

We have proven in class that a subset Z of $\mathbb{P}^n \times \mathbb{P}^m$ is closed, in the Zariski topology of the product variety (induced by the Segre embedding), if and only if $Z = V(S)$, for some set S of bi-homogeneous polynomials.

[1] This means that $K(X)$ is an extension of k of transcendence degree ≥ 1. All you need here is that X is not a point.

(a) Let $A_{++} \subset A$ be the ideal generated by all the products X_iY_j, $0 \leq i \leq n$, $0 \leq j \leq m$. Prove the bi-homogeneous Nullstellensatz: There is a one-to-one order reversing correspondence between radical bi-homogeneous ideals not containing A_{++}, and non-empty closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$. Hint: Immitate the proof that affine Nallstellensatz implies projective Nallstellensatz. Note that the subset $V_{\text{affine}}(A_{++})$ of $\mathbb{A}^{n+m+2}$ is the union $\mathbb{A}^{n+1} \times \{0\} \cup \{0\} \times \mathbb{A}^{m+1}$. Use the linear algebra fact, that an ideal of A is bi-homogeneous, if and only if it is $(k^* \times k^*)$-invariant.

(b) Assume that $V(S) \neq \emptyset$. Show that $V(S)$ is irreducible, if and only if the radical of the bi-homogeneous ideal generated by S is a prime ideal in A.

(c) Let $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the Segre embedding,

$$\varphi[(x_0, x_1), (y_0, y_1)] = (x_0y_0, x_0y_1, x_1y_0, x_1y_1).$$

Let the homogeneous coordinates on $\mathbb{P}^3$ be X, Y, Z, W, so that the image of φ is $V(XW - YZ)$.

i. Find a bi-homogeneous polynomial $F(X_0, X_1, Y_0, Y_1)$, such that $\varphi(V(F)) = V(XW - YZ,\ X^2 + Y^2 + Z^2 + W^2)$. Show that $V(F)$ is the union of four irreducible components, each isomorphic to $\mathbb{P}^1$.

ii. Let $\rho : \mathbb{P}^1 \to \mathbb{P}^3$ be the twisted cubic, $\rho(s,t) = (s^3, s^2t, st^2, t^3)$. Find a bi-homogeneous polynomial $G(X_0, X_1, Y_0, Y_1)$, such that $\varphi^{-1}(\rho(\mathbb{P}^1)) = V(G)$. Compare with problem 5 of Homework 2.

6. (Shafarevich I.4.9 and I.4.10 modified) Let X be an irreducible quadric in $\mathbb{P}^3$ defined by a quadratic polynomial of maximal rank. Note that the introduction in HW3 Problem 7 and Problem 5c above imply that X is isomorphic to the image of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ via a linear change of coordinates of $\mathbb{P}^3$.

(a) Show that projection from a point of X determines a birational map $\varphi : X \to \mathbb{P}^2$. Compare with the last example in Lecture 8, where we showed that the projection from a point on an irreducible conic $C \subset \mathbb{P}^2$ is an isomorphism $\pi : C \to \mathbb{P}^1$.

(b) Determine the domain of regularity of φ and φ^{-1}.

(c) Find open sets $U \subset X$ and $V \subset \mathbb{P}^2$, which are isomorphic via φ.

7. (Shafarevich I.4.11) Let $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ be the rational map given by

$$\varphi(x_0 : x_1 : x_2) = (x_1x_2 : x_0x_2 : x_0x_1)$$

away from $V(x_1x_2, x_0x_2, x_0x_1)$. Show that φ is a birational map and find the domain of definition of φ^{-1}. Find open sets between which φ is an isomorphism.