Algebraic Geometry Homework Assignment 3, Spring 2021

The field k below is assumed algebraically closed.

(1) (Hartshorne, Exercise I.3.2, two examples, that a morphism, whose underlying map on the topological spaces is a homeomorphism, need not be an isomorphism).
 (a) Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a morphism and a homeomorphism (bijective) from $\mathbb{A}^1$ onto $V(y^2 - x^3)$, but that φ is not an isomorphism.
 (b) Let the characteristic of k be a prime $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that the morphism φ is a homeomorphism, but not an isomorphism. This is called the *Frobenius morphism*.

(2) (Shafarevich, Problem I.2.7) Show that the hyperbola $V(xy - 1) \subset \mathbb{A}^2$ is not isomorphic to $\mathbb{A}^1$.

(3) (Shafarevich, Exercise I.2.8) Consider the image $S := f(\mathbb{A}^2)$ of the morphism $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $f(x, y) = (x, xy)$. Is S open in $\mathbb{A}^2$? Is it dense? Is it closed?

(4) (Shafarevich, Exercise I.2.10) Show that all automorphisms of $\mathbb{A}^1$ (isomorphisms from $\mathbb{A}^1$ onto itself) are of the form $f(x) = ax + b$, $a \neq 0$.

(5) (Shafarevich, Exercise I.2.11) Let $P(x)$ be an arbitrary polynomial in x. Show that the map $f : \mathbb{A}^2 \to \mathbb{A}^2$, given by $f(x, y) = (x, y + P(x))$ is an automorphism of $\mathbb{A}^2$.

(6) (Hartshorne, Exercise I.3.6, an example of a quasi-affine variety, which is not affine). Show that $X := \mathbb{A}^2 \setminus \{(0, 0)\}$ is not affine. Hint: Show that $\Gamma(X) \cong k[x, y]$ and use the equivalence between the categories of affine varieties and that of finitely generated k-algebras which are integral domains.

(7) The following problem was touched upon at the end of Lecture 4. Assume that the characteristic $char(k)$ is different from 2. Let $f \in k[x_0, \dots, x_n]$ be a homogeneous polynomial of degree 2. By the theory of symmetric bilinear forms, there is a linear change of variables, which brings f to the form $x_0^2 + \cdots + x_k^2$, for some $0 \leq k \leq n$ (see Hoffman and Kunze, *Linear Algebra*, for example).
 (a) Show that f is irreducible, if and only if $k \geq 2$.
 (b) Show that after a linear change of coordinates, every plane conic (i.e., $V(f) \subset \mathbb{P}^2$, where f is irreducible, of degree 2, and $n = 2$) can be realized as the image $V(xz - y^2)$ of the 2-uple embedding $\phi : \mathbb{P}^1 \to \mathbb{P}^2$, given by $(s, t) \mapsto (s^2, st, t^2)$ (see Homework 2 Problem 7).
 (c) Construct an embedding $e : PGL(2) \to PGL(3)$, obtaining an action of $PGL(2)$ on $\mathbb{P}^2$, with respect to which the map ϕ is $PGL(2)$-equivariant, i.e., such that $\phi(g(s, t)) = e(g)\phi(s, t)$, for all $(s, t) \in \mathbb{P}^1$.
 (d) Let $C := V(f) \subset \mathbb{P}^2$ be an irreducible conic and $P = (a_0, a_1, a_2)$ a point in C. Let f_x be the partial $\frac{\partial f}{\partial x}$. Show that the line

$$f_x(P)x + f_y(P)y + f_z(P)z = 0$$

 intersects C at the point P and at no other point, and that any other line in $\mathbb{P}^2$ through P intersects C at precisely one additional point. Hint: $PGL(2)$ acts (triply) transitively on $\mathbb{P}^1$, so the statement reduces to the case $f(x, y, z) = xz - y^2$ and $P = (1, 0, 0)$.

(8) (Hartshorne, Exercise I.3.11 modified) Let X be an affine variety, $P \in X$ a point, and $m_P \subset \Gamma(X)$ its maximal ideal. Show that there is a one-to-one correspondence between the prime ideals of $\Gamma(X)_{m_p}$ and closed subvarieties of X containing P. Conclude, in particular, that $\Gamma(X)_{m_p}$ has a unique maximal ideal.

(9) (Filling in the proof of a statement from Lecture 6 about the coordinate ring of principal affine subsets) Let R be a commutative ring with 1, $f \in R \setminus \{0\}$, $S := \{f^n \ : \ n \geq 0\}$, and $R_f := S^{-1}R$.

(a) Set $A := R[y]/(yf - 1)$, where y is an indeterminate, and let $\phi : R \to A$ be the homomorphism $\phi(r) = r + (yf - 1)$. Prove that $\phi(r) = 0$, if and only if $rf^n = 0$, for some $n \geq 0$. Hint for the "only if" direction: If $\phi(r) = 0$, then there is a polynomial $g(y) = \sum_{i=0}^{n} c_i y^i$, such that $r = g(y)(yf - 1)$. Use the latter equality to solve for the coefficients $c_i \in R$.

(b) Let $h : R_f \to A$ be the natural homomorphism, which is determined by the universal property of R_f and sends r/f^n to $\phi(r)y^n$. Prove that h is an isomorphism.

(10) (Shafarevich, Exercise I.4.14) Let X be an irreducible conic in $\mathbb{P}^2$. Show that the quasiprojective variety $\mathbb{P}^2 \setminus X$ is affine. Hint: Use the fact the the 2-uple embedding $\rho_2 : \mathbb{P}^2 \to \mathbb{P}^5$ from Q7 in HW2 (called the Veronese embedding in Shafarevich) is an isomorphism onto its image.