Algebraic Geometry Math 797 Homework Assignment 2, Fall 2021

The field k below is assumed algebraically closed.

1. (Hartshorne I.1.2, the affine *twisted cubic curve* revisited) Let $Y \subset \mathbb{A}^3$ be the set $\{(t, t^2, t^3) \ : \ t \in k\}$. Find generators for $I(Y)$ and show that its affine coordinate ring $k[x, y, z]/I(Y)$ is isomorphic to a polynomial ring in one variable over k.

2. (Hartshorne I.1.3) Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is the union of three irreducible components and find their prime ideals.

3. (Hartshorne I.1.4) Identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way. Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topology on the two copies of $\mathbb{A}^1$.

4. (Hartshorne I.1.6) Any non-empty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

5. (Mumford, section I.2 Example B) Let $\phi : \mathbb{P}^1 \to \mathbb{P}^3$ be the parametrization of the twisted cubic curve
$$\phi(s, t) \quad = \quad (s^3, s^2t, st^2, t^3),$$
and $\phi^* : k[x, y, z, w] \to k[s, t]$ the pullback homomorphism, $\phi^*(x) = s^3$, $\phi^*(y) = s^2t$, $\phi^*(z) = st^2$, $\phi^*(w) = t^3$. We have seen in class, that the image $C := \phi(\mathbb{P}^1)$ is cut out by the homogeneous ideal $J := (xz - y^2, yw - z^2, xw - yz)$. Conclude that $I(C) = \sqrt{J}$.

   (a) Prove that $J = \ker(\phi^*)$. Conclude that J is a prime ideal and $I(C) = J$. Hint: Reduce to the following statement. Given non-negative integers (a, b) and a polynomial $f \in \ker(\phi^*)$, which is a linear combination of monomials $M = x^{e_x}y^{e_y}z^{e_z}w^{e_w}$ with $\phi^*(M) = s^at^b$, then f belongs to J. We may assume $a \geq b$, by interchanging the roles of s and t. Treat the cases $a > b$ and $a = b$ separately. If $a > b$, prove it by a double induction, on $\deg(f)$ and the degree of f as a polynomial in y with coefficients in $k[x, z, w]$. If $\deg_y(f) \leq 1$, show that x divides f and use the induction hypothesis (and the fact that J is prime). If $\deg_y(f) \geq 2$, use the element $xz - y^2$ of J to lower $\deg_y(f)$. For the case $a = b$ you may need a triple induction.

   (b) Prove that the projective coordinate rings $k[s, t]$ of $\mathbb{P}^1$ and $k[x, y, z, w]/I(C)$, of the twisted cubic curve, are not isomorphic. (Contrast with the affine case in Question 1).

6. (Hartshorne I.2.9, *projective closure of an affine variety*) Let $(y_1, \ldots, y_n)$ be affine coordinates on $\mathbb{A}^n$, $(x_0, x_1, \ldots, x_n)$ homogeneous coordinates on $\mathbb{P}^n$, U_0 the complement of the hyperplane $x_0 = 0$, and identify $\mathbb{A}^n$ with U_0 via the natural homeomorphism $\varphi_0 : U_0 \to \mathbb{A}^n$, so that the y_i coordinate of $\varphi(x_0, x_1, \ldots, x_n)$ is x_i/x_0.

Given a polynomial $g \in k[y_1, \ldots, y_n]$ of degree d, set

$$\beta(g) = g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) x_0^d.$$

Let $Y \subset \mathbb{A}^n$ be an affine variety. The closure $\overline{Y}$ of Y in $\mathbb{P}^n$ is called its *projective closure*.

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$.

(b) Let $Y := V(g) \subset \mathbb{A}^n$ be a hypersurface associated to a non-constant square-free polynomial g. Show that $I(\overline{Y}) = (\beta(g))$. Hint: Show first that $I(Y) = (g)$.

(c) Let $Y \subset \mathbb{A}^3$ be the twisted cubic of question 1. Use your answer to question 5 in order to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

7. (Hartshorne I.2.12, *The d-Uple embedding*) For given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree d in the variables $x_0, \ldots, x_n$. Note that $N = \binom{n+d}{n} - 1$. Let $\rho_d : \mathbb{P}^n \to \mathbb{P}^N$ be given by

$$\rho_d(a) = (M_0(a), \ldots, M_n(a)),$$

where $a = (a_0, \ldots, a_n)$ is the set of homogeneous coordinates of a point. This is called the d-Uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1$, $d = 2$, then $N = 2$ and the image of the 2-Uple embedding of $\mathbb{P}^1$ is a conic in $\mathbb{P}^2$. Note also that if $M_0(x_0, \ldots, x_n) = x_0^d$, then ρ_d restricts to the affine open subset $\mathbb{P}^n_{x_0}$, where $x_0 \neq 0$, as the map $(m_1, \ldots, m_N)$ from $\mathbb{A}^n$ to $\mathbb{A}^N$, where the $m_i := M_i/M_0$ run through all monomials in $x_1/x_0, \ldots, x_n/x_0$ of degree $\leq d$.

(a) Let $\theta : k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homomorphism given by $\theta(y_i) = M_i$, and let J be the kernel of θ. Then J is a homogeneous prime ideal and so $V(J)$ is a projective variety in $\mathbb{P}^N$.

(b) Show that the image of ρ_d is contained in $V(J)$. Note: Hartshorne asks to prove that the image of ρ_d is equal to $V(J)$ and that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $V(J)$. This is combinatorially challenging. You need not prove it, but you should know it.

(c) Show that the twisted cubic curve of question 5 is equal to the 3-Uple embedding of $\mathbb{P}^1$, for suitable choice of coordinates.