Algebraic Geometry Math 797 Homework Assignment 1, Fall 2021

1. Prove that the set $C := \{(t, t^2, t^3) \ : \ t \in k\}$ is an algebraic subset of $\mathbb{A}^3$.

2. Let $I_1 = (x^2 + y, \ x)$ and $I_2 = (y^2x^2 + x^2 + y^3 + y + xy, \ yx^2 + y^2 + x)$. Show the equality of the algebraic subsets $V(I_1) = V(I_2)$ in $\mathbb{A}^2(\mathbb{Q})$, over the fields $\mathbb{Q}$ of rational numbers.

3. Let k be an algebraically closed field, X an algebraic subset of $\mathbb{A}^n(k)$, and P a point of $\mathbb{A}^n(k)$, which is not in X. Show that there is a polynomial F in $k[x_1, \ldots, x_n]$, such that $F(Q) = 0$, for all $Q \in X$, but $F(P) = 1$.

4. (a) If I_1 and I_2 are ideals of some commutative ring R, show that $\sqrt{I_1 I_2} = \sqrt{I_1 \cap I_2}$.

   (b) If I_1 and I_2 are radical ideals, show that $I_1 \cap I_2$ is a radical ideal.

5. Let k be algebraically closed, and $X \subset \mathbb{A}^3(k)$ the union of the x_1-axis and the point $(1, 1, 1)$. Find generators for $I(X)$.

6. Let k be a field of characteristic $\neq 2$. Prove that there are three points $a, b, c \in \mathbb{A}^2(k)$, such that

$$\sqrt{(x^2 - 2xy^4 + y^6, y^3 - y)} = \mathfrak{m}_a \cap \mathfrak{m}_b \cap \mathfrak{m}_c,$$

   where $\mathfrak{m}_a$ is the maximal ideal of the point a, etc...

7. Let k be an algebraically closed field and $I \subset k[x_1, \ldots, x_n]$ an ideal. Prove that $V(I)$ is a single point, if and only if $\sqrt{I}$ is a maximal ideal.

8. Let k be an algebraically closed field.

   (a) Show that the polynomial $y^2 - x(x-1)(x-\lambda)$ is irreducible, for every $\lambda \in k$. Hint: Use Eisenstein's Criterion, or otherwise.

   (b) Show also that the polynomial $y^2 - x^3$ is irreducible.

9. **Definitions**

   i Let $X \subset \mathbb{A}^n(k)$ be an affine algebraic subset. The *affine coordinate ring* of X is the ring $R := k[x_1, \ldots, x_n]/I(X)$.

   ii Let A be an integral domain and K its fraction field. Recall that the integral closure of A is the subring $\overline{A}$ of K, consisting of all elements of K, which are integral over A. A is said to be *integrally closed*, if $A = \overline{A}$.

(a) Let k be an algebraically closed field, R the coordinate ring of the affine cubic plane curve $V(Y^2 - X^3)$, and K the fraction field of R. Prove that R is not integrally closed, i.e., find an element of K, which is integral over R, but does not belong to R.
Notational suggestion: Denote the images of X and Y in R by x, y.

(b) Repeat part 9a, but with the nodal cubic curve $V(Y^2 - X^2(X - 1))$.

Note: We will later see, that an affine algebraic curve is smooth and connected (to be defined), if and only if its coordinate ring is integrally closed.