Homework 5

1. (Debbare, Ch. 6, Q5) Let $X = V/\Lambda$ be a compact complex torus and let $f : X \to \hat{X}$ be a homomorphism. Show that the following are equivalent

   (a) There exists a line bundle L on X, such that $f = \phi_L$ (i.e., $f(x) = (t_x^* L) \otimes L^{-1}$).

   (b) $u = \hat{u}$.

2. (Debbare, Ch. 6, Q8) Let D be an ample effective divisor on an abelian variety X. Show that any compact analytic subset Y of X that does not meet the support of D is finite. Hint: Y is contained in a fiber of the morphism $\varphi_{\mathcal{O}_X(3D)} : X \to |D|^*$.

3. (Birkenhake-Lange, Ch. 5 problem 1) For a square free integer $d \geq 1$ consider the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$ with maximal order R (the ring of integers, i.e., the integral closure of $\mathbb{Z}$). If $\{1, \omega\}$ denotes the usual basis of R and f is a positive integer, then $R_f := \mathbb{Z} \oplus f\omega\mathbb{Z}$ is a lattice in $\mathbb{C}$ and $E_f := \mathbb{C}/R_f$ is an elliptic curve. Show that $\mathrm{End}(E_f) = R_f$. In particular, if $f \geq 2$, then $\mathrm{End}(E_f)$ is not a maximal order in $\mathbb{Q}(\sqrt{-D})$.

4. (Birkenhake-Lange, Ch. 5 problem 2) Let X_j be an abelian variety with polarization L_j of degree d_j for $j = 1, 2$. Then $p_1^* L_1 \otimes p_2^* L_2$ is a polarization of $X_1 \times X_2$ of degree $d_1 d_2$.

5. (Birkenhake-Lange, Ch. 5 problem 3 - unreasonably hard as formulated - modified as in Debbare, Ch. 6, Q14). Let (X, H) be a polarized abelian variety, and let n be a non-zero integer such that $n \ker(\phi_H) = 0$, so that $n\phi_H^{-1}$ defines a homomorphism $\hat{X} \to X$.

   (a) Suppose there exists an endomorphism f of X satisfying
   $$ff' = f'f = (n-1)id_X.$$
   Then there exists a line bundle M on $X \times \hat{X}$, such that the matrix of $\phi_M : X \times \hat{X} \to \widehat{(X \times \hat{X})}$ is
   $$\begin{pmatrix} \phi_H & \hat{f} \\ f & n\phi_H^{-1} \end{pmatrix}$$
   by Problem 1 above. Show that the map ϕ_M is injective and that M defines a principal polarization on $X \times \hat{X}$.

   (b) Show that there exists a principal polarization on the abelian variety $(X \times \hat{X})^4$. Hint: Consider an endomorphism f of X^4 with matrix
   $$\begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ a & c & -b & a \end{pmatrix}$$
   where a, b, c, d are suitably chosen integers. Note that the product of the above matrix with its transpose is $(a^2 + b^2 + c^2 + d^2)I$.

6. (Debbare, Ch. 6 Q15) Let X be an abelian variety such that $\mathrm{End}(X)$ is isomorphic to $\mathbb{Z}$.

   (a) Show that any abelian variety isogenous to X has the same property.

   (b) Show that X is simple.

   (c) Show that the Neron-Severi group $NS(X)$ of X is isomorphic to $\mathbb{Z}$. In particular, the polarizations of X are all proportional.

7. (Birkenhake-Lange, Ch. 4 problem 10 modified - this beautiful problem is unreasonably hard as originally formulated as the solution depends on later material in Section 5.4 of Birkenhake-Lange). Let X be a simple abelian variety. Show that any algebraic subvarieties V and W of X with $\dim(V)+\dim(W) \geq \dim(X)$ have a non-empty intersection. Hint: Reduce to the case where the two subvarieties have complementary dimensions. Theorem 4.9.4 in Birkenhake-Lange and the moving Lemma [Birkenhake-Lange, Lemma 5.4.1] enable one to define the algebraic endomorphism $\delta(V, W) : X \to X$ sending $x \in X$ to the sum of the points of intersection of V with $t_x^*(W)$, for x in the Zariski open subset where the intersection is transversal.