Homework 4

1. (Birkenhake-Lange, Ch. 2 problem 2, First Chern class of a line bundle) Let $X = V/\Lambda$ be a compact complex torus. Recall that the first Chern class is the image of the class of L via the connecting homomorphism $c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$ of the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \overset{2\pi i(\bullet)}{\to} \mathcal{O}_X^* \to 0$. Suppose that L is given by the 1-cocycle $\{f_{ij}\}$ in $Z^1(\mathcal{U}, \mathcal{O}_X^*)$ associated to the canonical factor $a_{H,\chi} \in Z^1(\Lambda, H^0(\mathcal{O}_V^*))$ of L via the homomorphism ϕ_1 in Appendix B of Birkenhake-Lange discussed in class. So $\pi : V \to X$ is the universal covering and $\mathcal{U} := \{U_i\}_{i\in I}$ is a sufficiently fine open covering of X, such the index i determines also a connected component $\widetilde{U}_i$ of $\pi^{-1}(U_i)$ and the restriction $\pi_i : \widetilde{U}_i \to U_i$ of π is a homeomorphism. The cocycle is given by $f_{ij} = a_{H,\chi}(\lambda_{ij}, \pi_i^{-1}{}_{|U_{ij}}(\bullet))$ where λ_{ij} is the unique element of Λ, such that $(\lambda_{ij} + \widetilde{U}_i) \cap \widetilde{U}_j$ is non-empty. The image of $c_1(L)$ in $H^2_{\mathrm{DR}}(X)$ is then given as follows. Choose C^∞ 1-forms φ_i on U_i, such that $\frac{1}{2\pi i} d\log(f_{ij}) = \varphi_j - \varphi_i$. Then $c_1(L)$ is the class of the global 2-form $(d\varphi_i)_{i\in I}$ (a 0-cocycle).

   (a) Let $\{v_1, \ldots, v_g\}$ be complex coordinates on V with respect to a basis $\{e_1, \ldots, e_g\}$ of V. Show that

   $$c_1(L) = \frac{i}{2} \sum_{j=1}^{g} \sum_{k=1}^{g} H(e_j, e_k) dv_j \wedge d\bar{v}_k. \tag{1}$$

   Note that the right hand side above is $\frac{i}{2}\partial\bar{\partial}h(v)$, where $h(v) = H(v, v)$, as $H(v, w)$ is linear (hence holomorphic) in v and anti-linear (hence anti-holomorphic) in w, so $\bar{\partial}h(v) = \sum_{j=1}^{g} H(v, e_j) d\bar{v}_j$.

   (b) Suppose that L is of type $(d_1, d_2, \ldots, d_g)$. Let $\{\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g\}$ be a symplectic basis of Λ for L and let $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$ be the corresponding real coordinates on V. Show that

   $$c_1(L) = -\sum_{j=1}^{g} d_j dx_j \wedge dy_j. \tag{2}$$

   Hint: Equation (1) reduces the above equation to the equality

   $$\frac{i}{2}(\partial\bar{\partial}h) = -\sum_{j=1}^{g} ImH\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\right) dx_j \wedge dy_j, \tag{3}$$

   where we identified the real tangent bundle of X with the trivial bundle with fiber V and regard global translation invariant vector fields as elements of V (so λ_j corresponds to $\frac{\partial}{\partial x_j}$ etc...). The operators ∂ and $\bar{\partial}$ depend only on the complex structure. Equation (3) can be stated in a coordinate free way as the equality

   $$\frac{i}{2}(\partial\bar{\partial}h)(\xi_1, \xi_2) = -ImH(\xi_1, \xi_2),$$

for every two translation invariant vector fields ξ_1, ξ_2. Prove the latter equality using coordinates $\tilde{v}_j = \tilde{x}_j + i\tilde{y}_j$, $1 \leq j \leq g$, for V with respect to an orthogonal basis $\{\tilde{e}_1, \dots, \tilde{e}_g\}$ of V with respect to H.

2. (Birkenhake-Lange, Ch. 4 problem 1) Suppose X is an abelian variety with period matrix $\Pi \in M(g \times 2g, \mathbb{C})$ and $A \in M(2g \times 2g, \mathbb{Z})$ is the alternating matrix defining a polarization as in the statement of the Riemann bilinear conditions [Birkenhake-Lange, Theorem 4.2.1]. There is a matrix $\Lambda \in M(g \times 2g, \mathbb{C})$, such that $A = (\Pi^t)\Lambda - (\Lambda^t)\Pi$. Hint: Reduce to the case where $A = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ is the matrix with respect to a symplectic basis for the polarization and $\Pi = (\Pi_1 I_g)$. Recall that in this case the Riemann bilinear conditions state that $Z := \Pi_1 D^{-1}$ is symmetric with a positive definite imaginary part.

3. (Birkenhake-Lange, Ch. 4 problem 2) Suppose that X and X' are abelian varieties with period matrices $\Pi \in M(g \times 2g, \mathbb{C})$ and $\Pi' \in M(g' \times 2g', \mathbb{C})$. There is a non-trivial homomorphism $q : X \to X'$ if and only if there is a non-zero matrix $Q \in M(2g' \times 2g)$ with $\Pi' Q \Pi^t = 0$. Hint: Use Question 2.

4. (Birkenhake-Lange, Ch. 4 problem 8)

   (a) Show that if $(Z, \mathbf{1}_g)$ is the period matrix of a complex torus, then $(Z^t, \mathbf{1}_g)$ is a period matrix of $\hat{X}$.

   (b) Conclude that there exists a complex torus X not isogenous to its dual $\hat{X}$.

5. (Birkenhake-Lange, Ch. 4 problem 4) Let $X = V/\Lambda$ be an abelian variety of dimension g and D a reduced effective divisor on X. Show that the Gauss map $G : D_s \to \mathbb{P}_{g-1}$ is given as follows. If $z_1, \dots, z_g$ denote complex coordinate functions on V and ω_j is the restriction to D of the translation invariant form

$$dz_1 \wedge \ldots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \ldots \wedge dz_g,$$

   $1 \leq j \leq g$, then

$$G(\bar{v}) = (\omega_1(v) : \cdots : \omega_g(v)),$$

   for every $\bar{v} \in D_s$ with representative $v \in V$. Above $\omega_j(v)$ is considered as the value at v of the section of the canonical line-bundle $\Omega^{g-1}_{\tilde{D}_s}$ on the zero divisor $\tilde{D} \subset V$ of the theta function.

6. (Birkenhake-Lange, Ch. 4 problem 5) Let $X = E_1 \times \cdots \times E_g$ be a product of elliptic curves. Consider the divisor $D = \sum_{j=1}^{g} E_1 \times \cdots \times E_{j-1} \times \{0\} \times E_{j+1} \times \cdots \times E_g$. Show that the image of the Gauss map $G : D_s \to \mathbb{P}_{g-1}$ consists of g points spanning $\mathbb{P}_{g-1}$. In particular, the image is not contained in any hyperplane, in agreement with [Birkenhake-Lange, Proposition 4.4.1].