Homework 3

1. (A geometric interpretation of Theorem 3.2.7 and Corollary 3.2.9 in Birkenhake-Lange) Let $X = V/\Lambda$ be a compact complex torus.

   (a) Let $f, g : \Lambda \times V \to \mathbb{C}^*$ be two factors of automorphy in $Z^1(\Lambda, H^0(\mathcal{O}_V^*))$ with $[f] = [g]$ in $H^1(\Lambda, H^0(\mathcal{O}_V^*))$. Choose a 0-co-chain $h \in C^0(\Lambda, H^0(\mathcal{O}_V^*)) = H^0(V, \mathcal{O}_V^*)$, such that $f(\lambda, v) = g(\lambda, v)h(v+\lambda)h(v)^{-1}$, for all $(\lambda, v) \in \Lambda \times V$. Let $L \in \mathrm{Pic}(X)$ be the line bundle with class $[f]$ under the isomorphism $H^1(\Lambda, H^0(\mathcal{O}_V^*)) \cong \mathrm{Pic}(X)$. Recall that $H^0(X, L) \cong H^0(V, \mathcal{O}_V)^f$, where the latter is the space of global holomorphic functions on V which graph is invariant with respect to the action $\lambda \cdot (v, t) = (v + \lambda, f(\lambda, v)t)$ of Λ on the trivial line bundle $V \times \mathbb{C}$ with respect to the factor of automorphy f

   $$H^0(V, \mathcal{O}_V)^f := \{\theta \in H^0(V, \mathcal{O}_V) \ : \ \theta(v+\lambda) = f(\lambda, v)\theta(v)\}.$$

   Show that multiplication by h yields an isomorphism $H^0(V, \mathcal{O}_V)^g \to H^0(V, \mathcal{O}_V)^f$.

   (b) Given $c \in V$, $(id_\Lambda \times t_c)^*(f)$ is a factor of automorphy for $t_{\bar{c}}^* L$, where $\bar{c} := c + \Lambda$, $t_c(v) = v + c$, and $t_{\bar{c}}(x) = x + \bar{c}$. Show that pull-back by t_c yields an isomorphism

   $$t_c^* : H^0(V, \mathcal{O}_V)^f \to H^0(V, \mathcal{O}_V)^{(id_\Lambda \times t_c)^*(f)}.$$

   (c) Let $L = L(H, \chi)$ be a line bundle with canonical factor $a_L = a_{(H,\chi)}$. Show that

   $$\varphi : H^0(V, \mathcal{O}_V)^{a_L} \to H^0(V, \mathcal{O}_V)^{a_{t_{\bar{c}}^* L}}, \tag{1}$$

   given by $\varphi(\theta) = e^{-\pi H(\cdot, c)} t_c^*(\theta)$, is a well defined isomorphism. Hint: The canonical factor $a_{t_{\bar{c}}^* L}$ and the factor $t_c^* a_L$ differ by a coboundary, see the proof of Lemma 2.3.2 in Birkenhake-Lange. **Remark:** The isomorphism in Corollary 3.2.9 in Birkenhake-Lange is a scalar multiple of φ.

   (d) Conclude that if $\bar{w} = w + \Lambda$ belongs to $K(L) := \Lambda(L)/\Lambda$ and θ is in $H^0(V, \mathcal{O}_V)^{a_L}$, then $\theta_{\bar{w}} := a_L(w, \cdot)^{-1} t_w^*(\theta)$ belongs to $H^0(V, \mathcal{O}_V)^{a_L}$ and depends only on $\bar{w}$. Hence, the following is a subset of $H^0(V, \mathcal{O}_V)^{a_L}$

   $$\{\theta_{\bar{w}} \ : \ \bar{w} \in K(L)_1\}, \tag{2}$$

   where $\Lambda(L) = \Lambda(L)_1 \oplus \Lambda(L)_2$ is the decomposition induced by a decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ for L, $K(L)_1 := \Lambda(L)_1/\Lambda_1$, and

   $$a_L : [\Lambda(L)_1 \oplus \Lambda_2] \times V \to \mathbb{C}_1 \tag{3}$$

   is the extension of a_L in Remark 3.1.5 in Birkenhake-Lange. Indeed, $a_L(w, \cdot) := e^{\pi H(\cdot, w) + \frac{\pi}{2} H(w,w)}$ is a scalar multiple of $e^{\pi H(\cdot, w)}$.

   (e) The map (3) above is a factor of automorphy for a line bundle M_1 on $X_1 := X/K(L)_1$ such that $L \cong p_1^* M_1$ and so a better notation for this map is a_{M_1}. We explained in class that the map

   $$K(L)_1 \times H^0(V, \mathcal{O}_V)^{a_L} \to H^0(V, \mathcal{O}_V)^{a_L}, \tag{4}$$

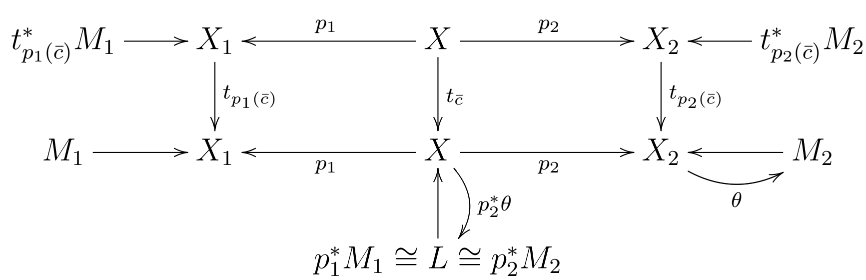
given by $(\bar{w}, \theta) \mapsto a_{M_1}(w, \cdot)^{-1} t_w^*(\theta)$, is the action of the Galois group $K(L)_1$ of $p_1 : X \to X_1$ induced by the action of the fundamental group $[\Lambda(L)_1 \oplus \Lambda_2]$ of X_1 on $H^0(V, \mathcal{O}_V)$ via the extension a_{M_1} of a_L. In this language (4) becomes:

$$Gal(X/X_1) \times H^0(X, p_1^* M_1) \to H^0(X, p_1^* M_1).$$

Use Theorem 3.2.7 of Birkenhake-Lange to show that $H^0(V, \mathcal{O}_V)^{a_L}$ is the regular representation of $K(L)_1$ via (4).

(f) Recall that $K(L)$ depends only on $E := c_1(L)$ and hence $K(L)_1$ depends only on $c_1(L)$ and the decomposition $\Lambda = \Lambda_1 \oplus \Lambda_2$ of Λ for E. Furthermore, $c_1(L) = c_1(t_{\bar{c}}^* L)$. Show that the isomorphism φ in (1) is an isomorphism of $K(L)_1$-representations.

(g) Set $X_2 := X/K(L)_2$, let $a_{M_2} : [\Lambda_1 \oplus \Lambda(L)_2] \times V \to \mathbb{C}_1$ be the analogue of a_{M_1} given in (3), and let M_2 be the corresponding line bundle on X_2. Show that the isomorphism φ in (1) maps the 1-dimensional subspace $H^0(V, \mathcal{O}_V)^{a_{M_2}}$ to $H^0(V, \mathcal{O}_V)^{a_{t^*_{p_2(\bar{c})} M_2}}$. When θ is a non-zero section of $H^0(V, \mathcal{O}_V)^{a_{M_2}}$, the subset (2) is a basis of $H^0(V, \mathcal{O}_V)^{a_L}$, by Corollary 3.2.5 and Theorem 3.2.7 of Birkenhake-Lange. Conclude that φ maps the basis (2) of $H^0(X, L)$, obtained as the set of $K(L)_1$-translates of a non-zero section of the 1-dimensional subspace $p_2^* H^0(X_2, M_2)$ of $H^0(X, p_2^* M_2) \cong H^2(X, L)$, to an analogous basis of $H^0(X, t_{\bar{c}}^* L)$.



2. (Birkenhake-Lange, Ch. 3 problem 3) With the notation of Section 3.2 show the following generalization of Corollary 3.2.5 of Birkenhake-Lange: For any $\bar{w} \in K(L)_1$ the function $\theta^c_{\bar{w}}$ is a canonical theta function for $t^*_{p_2(\bar{w})} M_2$, where M_2 is the descent to X_2 of the line bundle L as in Remark 3.1.5 of Birkenhake-Lange and $p_2 : X \to X_2$ is the quotient map. Show, furthermore, that $\theta^c_{\bar{w}}$ spans $H^0(V, \mathcal{O}_V)^{a_{t^*_{p_2(\bar{w})} M_2}} \cong H^0(X_2, t^*_{p_2(\bar{w})} M_2)$.

3. (Birkenhake-Lange, Ch. 3 problem 6, Mumford's Index Theorem) Let L_0 be a positive definite line bundle and L a non-degenerate line bundle on a compact complex torus $X = V/\Lambda$. Set $E_0 := c_1(L_0)$ and $E := c_1(L)$ considered as elements of $H^2(X, \mathbb{Z}) \cong Alt^2(\Lambda, \mathbb{Z})$. The Euler characteristic $\chi(L_0^n \otimes L)$ is equal to $(-1)^{s_n} f(n)$, where s_n is the index of $L_0^n \otimes L$ and $f(t) := Pf(tE_0 + E)$, by the Analytic Riemann-Roch Theorem 3.6.1 in Birkenhake-Lange. Prove the following statements.

(a) The polynomial $f(t)$ has only real roots.

(b) The index $i(L)$ is the number of positive roots of $f(t)$. Hint: Show that $f(-t)$ is the characteristic polynomial of $\phi_{H_0}^{-1} \circ \phi_H$, where H is the hermitian form associated to E and $\phi_H : V \to \bar{\Omega}$ is given by $\phi_H(v) = H(v, \bullet)$.

4. (Birkenhake-Lange, Ch. 3 problem 7) Let X be a compact complex torus of dimension g and let $\mathcal{P}$ be the Poincaré line bundle over $X \times \hat{X}$. Show that

$$h^q(\mathcal{P}) = \begin{cases} \mathbb{C} & \text{if} \quad q = g, \\ 0 & \text{if} \quad q \neq g. \end{cases}$$

Hint: Show that $\mathcal{P}$ is non-degenerate of index g.

5. (Birkenhake-Lange, Ch. 3 problem 8, Poincaré's Reducibility Theorem)

(a) Let X be a compact complex torus admitting a non-degenerate line bundle. For any complex subtorus Y of X such that $L|Y$ is non-degenerate, there exists a complex subtorus Z of X, such that $Y \cap Z$ is finite and $Y + Z = X$. In other words, X is isogenous to $Y \times Z$.

(b) Consider the complex tori $X = \mathbb{C}^2/\Pi\mathbb{Z}^4$, with

$$\Pi = \begin{pmatrix} i & \sqrt{2} & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}$$

and $Y = \mathbb{C}/(i,1)\mathbb{Z}^2$ embedded as a complex subtorus of X, where the embedding $Y \hookrightarrow X$ has analytic representation $z \mapsto (z, 0)$. Show that X does not admit any subtorus Z, such that $Y \times Z$ is isogenous to X.