HW3 Q5

Birkenhake - Lange Ch 3 Problem 8

(a) Let $L = L(H, \chi)$ be a non-degenerate l.b. on a complex torus $X = V/\Lambda$ and $Y \subset X$ a complex subtorus (and subgroup) such that $L_{|Y}$ is non-degenerate. So $Y = W/\Lambda_Y$, where $W \subset V$ is a complex subspace and $\Lambda_Y := \Lambda \cap W$ is a lattice of rank $2g'$, where $g' = \dim(Y) = \dim(W)$. (See HW 1 Q1).

Let $U \subset V$ be the subspace orthogonal to W w.r.t. H and set $\Lambda_Z := \Lambda \cap U$. Set $E := \operatorname{Im} H$.

Claim: $\Lambda_Z = \{\lambda \in \Lambda : E(\lambda, \Lambda_Y) = 0\}$.

Proof of the claim: ($\subseteq$) Clear, by def.

($\supseteq$) Let $\lambda \in \Lambda$ with $E(\lambda, \Lambda_Y) = (0)$ and $\mu \in \Lambda_Y$. Then

$$E(i\lambda, \mu) \underset{\uparrow}{=} E(i(i\lambda), i\mu) = -E(\lambda, i\mu).$$

Riemann bilinear cond (E is of type $(1,1)$)

Now $i\mu \in W = \operatorname{Span}_{\mathbb{R}} E(\Lambda_Y)$, and so $E(\lambda, \Lambda_Y) = (0) \Rightarrow E(\lambda, i\mu) = 0$.

Hence, $E(\lambda, \mu) = 0$ and $E(i\lambda, \mu) = 0$ and so $H(\lambda, \mu) := E(i\lambda, \mu) + i E(\lambda, \mu) = 0$. This is true for all $\mu \in \Lambda_Y$, and $\operatorname{Span}_{\mathbb{R}}(\Lambda_Y) = W$.

Hence $\lambda \in U$. So $\lambda \in \Lambda \cap U$. This proves the claim.

The claim implies that Λ_Z is a lattice in U. Hence $Z := U/\Lambda_Z$ is a (compact) complex subtorus of X.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
\Lambda_Y \oplus \Lambda_Z \arrow[r] \arrow[d] & \Lambda \arrow[d] \\
W \oplus U \arrow[r, "\cong", "i_W + i_U"'] \arrow[d, "(\pi_Y, \pi_Z)"'] & V \arrow[d, "\pi"] \\
W/\Lambda_Y \oplus U/\Lambda_Z \arrow[r, "i_Y + i_Z"] \arrow[d] & V/\Lambda \arrow[d] \\
0 & 0
\end{tikzcd}$$

So $\ker(i_Y + i_Z) \cong \Lambda/[\Lambda_Y + \Lambda_Z]$, by the snake Lemma. We have $\Lambda_Y \cap \Lambda_Z = (0)$, as H and (so E) restricts to W as a non-degenerate form.
Furthermore, $rk(\Lambda_Y) + rk(\Lambda_Z) = 2\dim_{\mathbb{C}}(W) + 2\dim_{\mathbb{C}}(U)$ $= 2\dim_{\mathbb{C}}(V) = rk(\Lambda)$. Hence, $\Lambda/[\Lambda_Y + \Lambda_Z]$ is finite and so is $\ker(i_Y + i_Z)$.

(b) $X = \mathbb{C}^2/\Pi\mathbb{Z}^4$, $\Pi = \begin{pmatrix} i & \sqrt{2} & 1 & 0 \\ 0 & i & 0 & 1 \end{pmatrix}$
$\{e_1, e_2\}$ the standard basis of $\mathbb{C}^2$,
$W = \operatorname{span}_{\mathbb{C}}\{e_1\}$, $\Lambda_Y = \operatorname{span}\left\{\begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

We show that $Y = W/\Lambda_Y$ is the unique subtorus (which is also a subgroup) of X.
It suffices to show that Λ_Y is the unique saturated non-zero proper sublattice of Λ which spans (over $\mathbb{R}$) a complex subspace, by HW1 Q1.

$(a_1, a_2, a_3, a_4)\Pi$ $\qquad$ $(b_1, b_2, b_3, b_4)\Pi$ $\qquad$ with $a_i, b_j \in \mathbb{Z}$

$$(+) \quad \left\{ \begin{pmatrix} a_1 i + a_2\sqrt{2} + a_3 \\ a_2 i + a_4 \end{pmatrix}, \begin{pmatrix} b_1 i + b_2\sqrt{2} + b_3 \\ b_2 i + b_4 \end{pmatrix} \right\}$$

span over $\mathbb{R}$ a 1-dim complex subspace $\Longrightarrow$

$$0 = (b_2 i + b_4)(a_1 i + a_2\sqrt{2} + a_3) - (a_2 i + a_4)(b_1 i + b_2\sqrt{2} + b_3)$$

$$= (a_3 b_4 - a_1 b_2) + a_2 b_4 \sqrt{2} + (a_1 b_4 + a_3 b_2) i + a_2 b_2 \sqrt{2}\, i$$

$$- (a_4 b_3 - b_1 a_2) - b_2 a_4 \sqrt{2} - (b_1 a_4 + b_3 a_2) i - b_2 a_2 \sqrt{2}\, i$$

Now, $\{1, \sqrt{2}, i, \sqrt{2}\, i\}$ are linearly indep / $\mathbb{Q}$.

Hence $\quad (a_3 b_4 - a_4 b_3) - (a_1 b_2 - b_1 a_2) = 0 \quad (1)$

$\qquad (a_2 b_4 - b_2 a_4) = 0 \quad (2)$

$\qquad (a_1 b_4 - b_1 a_4) + (a_3 b_2 - b_3 a_2) = 0 \quad (3)$

$\exists\, c_1, c_2 \in \mathbb{Q}$, not both zero, such that

$$c_1 \begin{pmatrix} a_2 \\ a_4 \end{pmatrix} + c_2 \begin{pmatrix} b_2 \\ b_4 \end{pmatrix} = 0, \text{ by } (2).$$

$\exists\, d_1, d_2 \in \mathbb{Q}$, not both zero, such that

$$d_1 \begin{pmatrix} a_1 \\ a_4 \end{pmatrix} + d_2 \begin{pmatrix} b_1 \\ b_4 \end{pmatrix} = 0, \text{ by } (3).$$

W.M.A that $c_1 \neq 0$, possibly after interchanging the roles of a_i's and b_j's. W.M.A $c_1 = 1$, $c_2 = c$.

$a_2 = -c b_2$, $\quad a_4 = -c b_4$. Eq (1) and (3) become

If $c_1 = 0$, then $b_2 = b_4 = 0$. So $a_4 = 0$.

Eq (1) becomes $b_1 a_2 = 0$. If $a_2 = 0$, then the subspace spanned by the two vectors (+) is

$$(a_3 b_4 + c b_4 b_3) - (a_1 b_2 + c b_1 b_2) = 0 \qquad (1')$$

$$\Leftrightarrow \quad \boxed{b_4(a_3 + cb_3) - b_2(a_1 + cb_1) = 0}$$

$$(a_1 b_4 + c b_1 b_4) + (a_3 b_2 + c b_3 b_2) = 0 \qquad (3')$$

$$\Leftrightarrow \quad \boxed{b_4(a_1 + cb_1) + b_2(a_3 + cb_3) = 0}$$

So $\begin{pmatrix} b_2 \\ b_4 \end{pmatrix}, \begin{pmatrix} a_3 + cb_3 \\ a_1 + cb_1 \end{pmatrix}$ are lin dep

and $\begin{pmatrix} b_2 \\ b_4 \end{pmatrix}, \begin{pmatrix} a_1 + cb_1 \\ -a_3 - cb_3 \end{pmatrix}$ are lin dep.

If $\begin{pmatrix} b_2 \\ b_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then the sublattice spanned by the two vectors $(+)$ is contained in Λ_y and we are done. If $\begin{pmatrix} b_2 \\ b_4 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then $\left\{ \begin{pmatrix} a_3 + cb_3 \\ a_1 + cb_1 \end{pmatrix}, \begin{pmatrix} a_1 + cb_1 \\ -a_3 - cb_3 \end{pmatrix} \right\}$ are linearly dependent. So $(a_3 + cb_3)^2 + (a_1 + cb_1)^2 = 0$.

So $(a_1 + cb_1) = 0 = (a_3 + cb_3)$.

So $(a_1, a_2, a_3, a_4) = (-c)(b_1, b_2, b_3, b_4)$.

So the two vectors in $(+)$ are linearly dependent over $\mathbb{C}$. So they do not span over $\mathbb{R}$ a complex subspace. ⌟