Homework 2

1. (Birkenhake-Lange, Ch. 2 problem 6, *Theorem of the cube*) Let X_k, $k = 1, 2, 3$, be complex tori and L a line bundle on $X_1 \times X_2 \times X_3$, such that the restrictions of L to $X_1 \times X_2 \times \{0\}$, $X_1 \times \{0\} \times X_3$, and $\{0\} \times X_2 \times X_3$ are trivial. Use canonical factors to prove that L is trivial. Note: The theorem of the cube holds also when X_i, $i = 1, 2, 3$, are compact complex manifolds, or projective varieties, or complete varieties, and is proved in this generality in Mumfords book *Abelian varieties.*

2. (Birkenhake-Lange, Ch. 2 problem 12)

   (a) Let X be a compact complex torus. Denote by $\Delta_X : X \to X \times X$ the diagonal embedding given by $\Delta_X(x) = (x, x)$. Let $\mu : X \times X \to X$ be the addition map, $\mu(x_1, x_2) = x_1 + x_2$. Prove the equality $\hat{\mu} = \Delta_{\hat{X}}$.

   (b) Let $f, g : X \to Y$ be homomorphisms of compact complex tori. Use part (2a) to show that $\widehat{(f + g)} = \hat{f} + \hat{g}$. You may assume Birkenhake-Lange, Ch. 2 problem 11 that the functor ˆ behaves well with respect to cartesian products.

3. (Birkenhake-Lange, Ch. 2 problem 15) Let X be a compact complex torus. Show that the pair $(\hat{X}, \mathcal{P})$ is uniquely determined (up to isomorphism) by the defining properties in the definition of the Poincaré line bundle $\mathcal{P}$ and by the universal property of $\mathcal{P}$ in Proposition 5.2 in Ch. 2 of Birkenhake-Lange. Explicitly, show that given a normal complex analytic space Y and a line bundle $\mathcal{P}'$ over $X \times Y$ satisfying:

   (a) The restriction of $\mathcal{P}'$ to $X \times \{y\}$ belongs to $\mathrm{Pic}^0(X)$.

   (b) The restriction of $\mathcal{P}'$ to $\{0\} \times Y$ is the trivial line bundle.

   (c) For any normal analytic space T and any line bundle $\mathcal{L}$ on $X \times T$ satisfying properties (3a) and (3b) above, there exists a unique holomorphic map $f : T \to Y$, such that $(id \times f)^* \mathcal{P}' \cong \mathcal{L}$.

   Show that there exists a biholomorphic map $f : \hat{X} \to Y$, such that $(id \times f)^* \mathcal{P}' \cong \mathcal{P}$.

4. (Birkenhake-Lange, Ch. 2 problem 16) Let $X = V/\Lambda$ be a compact complex torus and set $\overline{\Omega} := \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$. Given $v \in V$ let $ev_v : \overline{\Omega} \to \mathbb{C}$ be the evaluation at v, given by $ev_v(\ell) = \ell(v)$. The map $\tilde{\kappa} : V \to \mathrm{Hom}_{\overline{\mathbb{C}}}(\mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C}), \mathbb{C})$ given by $\tilde{\kappa}(v) = \overline{ev_v}$ is a $\mathbb{C}$-linear isomorphism called the *double antiduality isomorphism.*Let $\kappa : X \to \hat{\hat{X}}$ be the canonical isomorphism (with analytic representation $\tilde{\kappa}$). Denote by $\mathcal{P}_X$ the Poincaré line bundle for X and by $\mathcal{P}_{\hat{X}}$ the one for $\hat{X}$. Let

   $$s : \hat{X} \times X \to X \times \hat{X}$$

   be given by $s(\hat{x}, x) = (x, \hat{x})$. Show that $(1_{\hat{X}} \times \kappa)^* \mathcal{P}_{\hat{X}}$ is isomorphic to $s^* \mathcal{P}_X$ on $\hat{X} \times X$.

Remark: Birkenhake-Lange do not specify the double antiduality isomorphism $\tilde{\kappa}$. The above formula for $\tilde{\kappa}$ works so that the isomorphism κ has the desired property. Note however that there is a subtle sign choice involved for this property to hold. The price that we pay for this choice is that given $\lambda \in \Lambda$ and $\hat{\mu} \in \hat{\Lambda}$, one has

$$\langle \tilde{\kappa}(\lambda), \hat{\mu} \rangle := Im(\overline{ev_\lambda}(\hat{\mu})) = -Im(\hat{\mu}(\lambda)) = -\langle \hat{\mu}, \lambda \rangle.$$

Above we regard $\hat{\Lambda}$ as the sublattice of $\overline{\Omega}$ defined by $\{\ell \in \overline{\Omega} \ : \ Im(\ell(\Lambda)) \subset \mathbb{Z}\}$. Similarly, $\hat{\hat{\Lambda}}$ is the sublattice of $\mathrm{Hom}_{\overline{\mathbb{C}}}(\overline{\Omega}, \mathbb{C})$ given by $\{\ell \ : \ Im(\ell(\hat{\Lambda})) \subset \mathbb{Z}\}$ and the real pairing between V and $\overline{\Omega}$, given by $\langle \ell, v \rangle := Im(\ell(v))$, is the one defined in Section 2.2.4 of Birkenhake-Lange. So the pairings between Λ and $\hat{\Lambda}$ and between $\hat{\Lambda}$ and $\hat{\hat{\Lambda}}$ are compatible with respect to $-\tilde{\kappa}$ rather than $\tilde{\kappa}$. See Remark 9.12 in Huybrechts' book *Fourier-Mukai transforms in Algebraic Geometry.*

5. (Birkenhake-Lange, Ch. 2 problem 17) Let X be a compact complex torus and $\mathcal{P}$ the poincaré line bundle on $X \times \hat{X}$. Denote by p_1 and p_2 the projections and by $\mu : X \times X \to X$ the addition map.
   (a) Show that for any line bundle L on X, $(1_X \times \phi_L)^*\mathcal{P} \cong \mu^*L \otimes p_1^*L^{-1} \otimes p_2^*L^{-1}$.
   (b) Conclude that $c_1(L) = 0$, if and only if $\mu^*L \cong p_1^*L \otimes p_2^*L$.