HW 2 Problem 1 (B-Lange ch 2 Prob 6)

Let X_k, $k=1,2,3$, be compact cpx tori and L a l.b. on $X_1 \times X_2 \times X_3$ satisfying
$L|_{X_1 \times X_2 \times \{0\}}$, $L|_{X_1 \times \{0\} \times X_2}$, and $L|_{\{0\} \times X_1 \times X_2}$
are all trivial. Then L is trivial.

Proof: Write $X_k = V_k / \Lambda_k$. Then $L = L(H, \chi)$ where H is a hermitian form on $V_1 \oplus V_2 \oplus V_3$ and χ is a semi-character for H.
Set $E := \operatorname{Im} H$. Then $E \in \operatorname{Alt}^2(\Lambda, \mathbb{Z})$, $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$ and $\operatorname{Alt}^2(\Lambda, \mathbb{Z}) \subset \operatorname{Alt}^2(V, \mathbb{C})$; $V = V_1 \oplus V_2 \oplus V_3$
If $v = v_1 + v_2 + v_3$ and $u = u_1 + u_2 + u_3$, with $v_i, u_i \in V_i$,
then $E(v_1+v_2+v_3, u_1+u_2+u_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} E(v_i, u_j)$

and $E(v_i, u_j) = 0$ since $E|_{V_i + V_j} = 0$. So $E = 0$.
Hence, $H=0$. Thus, $\chi : \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3 \to \mathbb{C}_1$ is a character satisfying $\chi|_{\Lambda_i} \equiv 1$, $i=1,2,3$. It follows that for $\lambda_i \in \Lambda_i$, $i=1,2,3$
$\chi(\lambda_1 + \lambda_2 + \lambda_3) = \chi(\lambda_1)\chi(\lambda_2)\chi(\lambda_3) = 1 \cdot 1 \cdot 1 = 1$. Hence, χ is trivial. Thus, $L = L(0,1) = \mathcal{O}_{X_1 \times X_2 \times X_3}$ is the trivial l.b.

$\square$

HW2 Problem 2 (B-Lange Ch. 2 Problem 12)

a) $\Delta : X \to X \times X$ the diagonal map,
$\mu : X \times X \to X$ " addition. Then $\widehat{\mu} = \Delta_{\widehat{X}}$

proof:

Both Δ and μ are homomorphisms, so it suffices to prove that the analytic representations of $\widehat{\mu}$ and $\Delta_{\widehat{X}}$ are equal, by the uniqueness of the analytic representation. The analytic representation of $\Delta_{\widehat{X}}$ is the diagonal embedding $\Delta_{\overline{\Omega}} : \overline{\Omega} \to \overline{\Omega} \oplus \overline{\Omega}$. The analytic representation of μ is the addition $\mu_V : V \times V \to V$, given by $\mu_V(v_1, v_2) = v_1 + v_2$.
The analytic representation of $\widehat{\mu}$ is
$\mu_V^* : \underset{= \mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})}{\overline{\Omega}} \to \overline{\Omega} \oplus \overline{\Omega}$, given by

$\mu_V^*(l) = l \circ \mu_V$. Hence,

$\left(\mu_V^*(l)\right)(v_1, v_2) = l\left(\mu_V(v_1, v_2)\right) = l(v_1 + v_2) = l(v_1) + l(v_2) = \left(\Delta_{\overline{\Omega}}(l)\right)(v_1, v_2)$. We see that $\mu_V^* = \Delta_{\overline{\Omega}}$.

Hence, $\widehat{\mu} = \Delta_{\widehat{X}}$. $\square$

b) Let $f, g : X \to Y$ be homomorphisms. We have

$$\widehat{(f+g)} = \widehat{\mu \circ (f \times g)} \underset{\substack{\uparrow \\ \text{by functoriality} \\ \text{of } \widehat{\ }}}{=} \widehat{(f \times g)} \circ \widehat{\mu} \underset{\substack{\uparrow \\ \text{ch. 2 Prob 11 and part (a)}}}{=} (\widehat{f} \times \widehat{g}) \circ \Delta_{\widehat{X}} = \widehat{f} + \widehat{g}.$$

HW2 Problem 3 (B-Lange Prob 15 in ch 2)

Let Y be a normal complex analytic space and let $\mathcal{P}'$ be a line bundle on $X \times Y$ satisfying

(a) The restriction of $\mathcal{P}'$ to $X \times \{y\}$ has vanishing first Chern class

(b) $\mathcal{P}' | \{0\} \times Y \simeq \mathcal{O}_Y$ (the trivial line bundle)

(c) For any normal analytic space T and any l.b. $\mathcal{L}$ on $X \times T$ satisfying (a) and (b), there exists a unique holomorphic map $f: T \to Y$ such that $(id \times f)^* \mathcal{P}' \simeq \mathcal{L}$.

We need to show that there exists an isomorphism (biholomorphic map) $f: \hat{X} \to Y$, such that $(id \times f)^*(\mathcal{P}') \simeq \mathcal{P}$.

<u>Proof</u>; Property (c) and the defining property of the Poincaré l.b. $\mathcal{P}$ on $X \times \hat{X}$ implies that there exists a unique holo map $f: \hat{X} \to Y$, such that $(id \times f)^* \mathcal{P}' \simeq \mathcal{P}$.

Prop 5.2 in ch 2 of B-Lange implies that there exists a unique holo map $\psi: Y \to \hat{X}$, such that $(id \times \psi)^*(\mathcal{P}) \simeq \mathcal{P}'$, since $(Y, \mathcal{P}')$ satisfies properties (a) and (b) above.

We have $(id \times f \circ \psi)^*(\mathcal{P}') = (id \times \psi)^*((id \times f)^*(\mathcal{P}')) \simeq$
$\simeq (id \times \psi)^*(\mathcal{P}) \simeq \mathcal{P}'$. Hence, $f \circ \psi = id_Y$, by

the uniqueness of the map f in property (c) above. Similarly,

$$(id \times \psi \circ f)^*(\mathcal{P}) = (id \times f)^*\big((id \times \psi)^*(\mathcal{P})\big) \simeq (id \times f)^*(\mathcal{P}') \simeq \mathcal{P}.$$

Hence $\psi \circ f = id_{\hat{X}}$, by the uniqueness of the holo map ψ in Prop 5.2 of Ch 2 in [B-Lange]. We conclude that f is indeed biholomorphic. $\square$

HW 2 Problem 4 (B-Lange Ch 2. Problem 16)

Set $W := \mathrm{Hom}_{\overline{\mathbb{C}}}(\underbrace{\mathrm{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})}_{\overline{\Omega}}, \mathbb{C})$. Let $\tilde{\kappa} : V \to W$

be the double-anti-duality isomorphism given by $\tilde{\kappa}(v) = \overline{ev_v}$. Let

$$\tilde{s} : \overline{\Omega} \times V \to V \times \overline{\Omega}$$ be given by

$s(l, v) = (v, l)$. By definition $\tilde{\kappa}$ is the analytic representation of the isomorphism $\kappa : X \to \hat{\hat{X}}$. Clearly, $\tilde{s}$ is the analytic representation of the isomorphism

$$s : \hat{X} \times X \to X \times \hat{X}$$ given by $s(\hat{x}, x) = (x, \hat{x})$.

We will prove the isomorphism

$$(1_{\hat{X}} \times \kappa)^* \mathcal{P}_{\hat{X}} \cong s^* \mathcal{P}_X$$

by showing that their canonical factors are equal. The construction of $\mathcal{P}_{\hat{X}}$ in [B-Lange, Ch. 2, Th. 5.1] yields

$H_{\mathcal{P}_{\hat{X}}}((l_1, \omega_1), (l_2, \omega_2)) = \overline{\omega_2(l_1)} + \omega_1(l_2)$, for

$l_i \in \overline{\Omega}$ and $\omega_i \in W$. Hence

$$H_{(1_{\hat{X}} \times \kappa)^* \mathcal{P}_{\hat{X}}}((l_1, v_1), (l_2, v_2)) \overset{\text{by B-Lange, Ch. 2 Lemma 3.4}}{=} ev_{v_2}(l_1) + ev_{v_1}(l_2) =$$

$$= \overline{l_2(v_1)} + l_1(v_2) = H_{\mathcal{P}_X}((v_1, l_1), (v_2, l_2)) = H_{s^* \mathcal{P}_X}((l_1, v_1), (l_2, v_2)).$$

We conclude the equality $\boxed{H_{(1_{\hat{X}} \times \kappa)^* \mathcal{P}_{\hat{X}}} = H_{s^* \mathcal{P}_X}}$.

The semi-character for $H_{\mathcal{P}_{\hat{X}}}$ constructed in the proof of Theorem 5.1 in B-Lange Ch. 2 is

$\chi_{\mathcal{P}_{\hat{X}}} : \hat{\Lambda} \times \hat{\hat{\Lambda}} \longrightarrow \mathbb{C}_1$ given by

$$\chi_{\mathcal{P}_{\hat{X}}}(l, w) = e^{\pi i \operatorname{Im} w(l)} .$$

If $w = \tilde{\kappa}(v) = \overline{ev_v}$, then

$\operatorname{Im}(w(l)) = \operatorname{Im}\left(\overline{ev_v(l)}\right) = -\operatorname{Im} l(v) \in \mathbb{Z}$, for $l \in \hat{\Lambda}$ and $v \in \Lambda$.

Hence, $e^{\pi i \operatorname{Im}((\tilde{\kappa}(v))(l))} = e^{\pi i \operatorname{Im} l(v)}$. So

by B-Lange, Ch. 2 Lemma 3.4

$$\chi_{(1\times\kappa)^*\mathcal{P}_{\hat{X}}}(l, v) = e^{\pi i \operatorname{Im} l(v)} = \chi_{s^*\mathcal{P}_X}(l, v),$$

for $l \in \hat{\Lambda}$ and $v \in \Lambda$.

Since $\chi_{\mathcal{P}_X}(v, l) = e^{\pi i \operatorname{Im}(l(v))}$, by the proof of [B-Lange, Ch2 Th. 5.1]

We conclude that indeed, $(1\times\kappa)^*\mathcal{P}_{\hat{X}}$ and $s^*\mathcal{P}_X$ have the same canonical factors. $\square$

HW2 Problem 5 (B-Lange ch 2, Problem 17)

(a) $\mathcal{P}$ = Poincaré l.b. on $X \times \hat{X}$.
$\mu: X \times X \to X$ the addition map.
L a l.b. on X.
$\phi_L : X \to \hat{X}$, $\phi_L(x) = t_x^* L \otimes L^{-1}$.
we need to prove:

$$(1_X \times \phi_L)^* \mathcal{P} \simeq \underbrace{\mu^* L \otimes p_1^* L^{-2} \otimes p_2^* L^{-1}}_{M}.$$

The restriction of M to $X \times \{y\}$, $y \in \hat{X}$, is $t_y^* L \otimes L^{-1} = (1_X \times \phi_L)^* \mathcal{P}$. Indeed, if $e_y : X \to X \times X$ is given by $e_y(x) = (x, y)$, then

$$e_y^*(\mu^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}) \simeq \underbrace{(\mu \circ e_y)}_{t_y}{}^* L \otimes \underbrace{(p_1 \circ e_y)}_{id_X}{}^* L \otimes \underbrace{(p_2 \circ e_y)}_{\mathcal{O}_X}{}^* L$$

$$\simeq t_y^* L \otimes L^{-1}.$$

The restriction of M to $\{0\} \times X$ is $t_0^* L \otimes L^{-1} \simeq \mathcal{O}_X$ (= the trivial l.b.).

We conclude that $[(1_X \times \phi_L)^* \mathcal{P}] \otimes M^{-1}$ restricts to $X \times \{y\}$ as the trivial line bundle, for every $y \in X$, and it restricts to $\{0\} \times X$ as the trivial line-bundle. Hence, $(1_X \times \phi_L)^* \mathcal{P} \otimes M^{-1}$ is the trivial line bundle, by the Seesaw Theorem (A.8) in B-Lange.

(b) If $c_1(L)=0$, then $\phi_L : X \to \hat{X}$ is the constant 0 map, and so $(1_X \times \phi_L)^* \mathcal{P}$ is the restriction of $\mathcal{P}$ to $X \times \{0\}$, which is the trivial line bundle.

Hence $\mu^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1} \simeq \mathcal{O}_X$, by part (a). Thus $\mu^* L \underset{(*)}{\simeq} p_1^* L \otimes p_2^* L$.

Conversely, if $(*)$ holds, then $(1_X \times \phi_L)^* \mathcal{P}$ is the trivial line bundle, so its restriction to $X \times \{y\}$ is trivial for all $y \in X$. But the latter restriction is the restriction of $\mathcal{P}$ to $X \times \{\phi_L(y)\}$. Hence, $\phi_L(y) = 0$ $\forall y \in X$.

Hence, $L \in \mathrm{Pic}^0(X)$, by Lemma 4.7(b) in B-Lange Ch. 2.

$\square$