Homework 1

1. (Birkenhake-Lange, Ch. 1 problem 1) Let $X = V/\Lambda$ be a compact complex torus.

   (a) Show that X admits a complex subtorus of dimension g', if and only if there exists a subgroup $\Lambda' \subset \Lambda$ of rank $2g'$, such that the real subspace $\mathrm{span}_{\mathbb{R}}\{\Lambda'\}$ of V is a complex subspace.

   (b) Conclude that the set of complex subtori of X, which pass through $0 \in X$, is at most countable.

   (c) Give an example of a complex torus of dimension $g \geq 2$, which does not contain any complex subtorus of dimension g', $0 < g' < g$. Hint: Consider a period matrix of the form $\Pi = \begin{pmatrix} 1 & 0 & z_1 & z_3 \\ 0 & 1 & z_2 & z_4 \end{pmatrix}$, satisfying $\det\left(Im\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}\right) \neq 0$, $z_1 \notin \mathbb{Q}$, $z_2 \notin \mathbb{Q}(z_1)$, $z_3 \notin \mathbb{Q}(z_1, z_2)$, $z_4 \notin \mathbb{Q}(z_1, z_2, z_3)$.

2. (Birkenhake-Lange, Ch. 1 problem 2) Let $X = V/\Lambda$ be a compact complex torus of dimension g.

   (a) Show that there exists a basis $\{e_1, \ldots, e_g\}$ of V with respect to which the period matrix of X is of the form $(Z\ I_g)$, where I_g is the $g \times g$ identity matrix and Z is a $g \times g$ matrix with entries in $\mathbb{C}$ and $\det(Im(Z)) \neq 0$.

   (b) Prove the equality $\det\begin{pmatrix} Z & I \\ \bar{Z} & I \end{pmatrix} = \det(2iIm(Z))$.

3. (Birkenhake-Lange, Ch. 1 problem 6)

   (a) There is a bijection between the set of complex structures on $\mathbb{R}^{2g}$ and $GL_{2g}(\mathbb{R})/GL_g(\mathbb{C})$.

   (b) There is a bijection between the set of isomorphism classes of compact complex tori of dimension g and the set of orbits in $GL_{2g}(\mathbb{R})/GL_g(\mathbb{C})$ under the action of the subgroup $GL_{2g}(\mathbb{Z})$ of $GL_{2g}(\mathbb{R})$.

4. (Birkenhake-Lange, Ch. 1 problem 9) Let X be a compact complex torus and $\mu : X \times X \to X$ the addition map. Let $p_i : X \times X \to X$ be the projection onto the i-th factor. Show that a C^∞-one-form ω on X is translation invariant, if and only if $\mu^*\omega = p_1^*\omega + p_2^*\omega$.

5. (Birkenhake-Lange, Ch. 1 problem 10) Let Γ be a free abelian group of finite even rank. A *Hodge structure of weight* 1 *on* Γ is a decomposition of $\Gamma \otimes_{\mathbb{Z}} \mathbb{C}$ as a direct sum $H^{1,0} \oplus H^{0,1}$ of two complex subspace satisfying $H^{0,1} = \overline{H^{1,0}}$. Show that giving a Hodge structure on Γ is equivalent to giving a complex structure on the real torus $(\Gamma \otimes_{\mathbb{Z}} \mathbb{R})/\Gamma$, i.e., an isomorphism of real compact tori $(\Gamma \otimes_{\mathbb{Z}} \mathbb{R})/\Gamma \to X$ with a compact complex torus X.

6. (Birkenhake-Lange, Ch. 2 problem 4, the Néron-Severi group of a complex torus) Let (Z, I_g) be the period matrix of a compact complex torus X. We have seen that the Néron-Severi group $NS(X)$ of X is identified with the set of alternating forms on $\mathbb{R}^{2g}$, integer valued on $\mathbb{Z}^{2g}$, which induce hermetian forms on $\mathbb{C}^g$ via the $\mathbb{R}$-linear isomorphism $R^{2g} \to \mathbb{C}^g$ given by $x \mapsto (Z, I_g)x$. Let E be an alternating form on $\mathbb{R}^{2g}$ with matrix $M := \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$ in $M_{2g}(\mathbb{R})$. Show that the following conditions are equivalent.

   (a) E belongs to $NS(X)$.

   (b) $A, B, C \in M_g(\mathbb{Z})$ and

$$A - BZ + Z^tB^t + Z^tCZ = 0. \tag{1}$$

   Note that if $g = 2$ and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, $B = \begin{pmatrix} b & d \\ e & f \end{pmatrix}$, and $C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$, then condition (6b) reads: $a, b, c, d, e, f \in \mathbb{Z}$ and

$$a + ez_{1,1} - bz_{1,2} + fz_{2,1} - dz_{2,2} + c\det(Z) = 0. \tag{2}$$

   Hint: The columns of the period matrix (Z, I_g) form a basis β of $\mathbb{C}^g$ as a $2g$-dimensional vector space over $\mathbb{R}$. Let $J : \mathbb{C}^g \to \mathbb{C}^g$ be multiplication by i and let $R := [J]_\beta$ be the matrix of J in the basis β. Show that condition 6a above is equivalent to M having integer entries and $R^tMR = M$. Note that the latter equality is equivalent to MR being a symmetric matrix, since $R^{-1} = -R$. Now write $Z = X + iY$, with $X, Y \in M_g(\mathbb{R})$, and show that

$$R = \begin{pmatrix} Y^{-1}X & Y^{-1} \\ -[Y + XY^{-1}X] & -XY^{-1} \end{pmatrix}$$

   (argue as in the proof of Proposition 2.3 in Chapter 1 of Birkenhake-Lange). Finally show that the symmetry of MR is equivalent to equation (1).

   Notes: (i) Set $\mathfrak{T}_g := \{Z \in M_g(\mathbb{C}) \ : \ \det(Im(Z)) \neq 0\}$. Denote by X_Z the compact complex torus with period matrix (Z, I_g). The symmetry of MR provides real equations for the locus of Z in $\mathfrak{T}_g$ where E belongs to $NS(X_Z)$. The equivalent equation (1) shows that this locus is a complex subvariety.

   (ii) One can use Equation (2) to give an example of a two-dimensional compact complex torus with a trivial Néron-Severi group.

7. (Birkenhake-Lange, Ch. 2 problem 5) Let X be a compact complex torus of dimension g. The rank $\rho(X)$ of the Néron-Severi group of X is called the *Picard number* of X. Show that $\rho(X) \leq h^{1,1}(X) = g^2$.