HW 1 problem 1: $X = V/\Lambda$ a cpt cpx torus of dim g.
Let $\pi: V \to X$ be the quotient map.

(a) There exists an injective holomorphic map $f: Y \to X$ from a cpt cpx torus $Y = V_1'/\Lambda_1$ of dimension g'

$\Leftrightarrow$ There exists an injective homomorphism $f: Y \to X$ as above (by Ch 1, Prop 2.1 (b))

$\Leftrightarrow$ There exists an injective $\mathbb{C}$-linear map $F: V_1 \to V$ with $F(\Lambda_1) \subset \Lambda$ inducing the homomorphism f (by Ch 1, Prop 2.1 (c))

$\Leftrightarrow$ There exists a subgroup $\Lambda' \subset \Lambda$ such that $\mathrm{Span}_{\mathbb{R}}\{\Lambda'\}$ is a complex subspace of V.

The latter equivalence is proven as follows:

($\Rightarrow$) Take $\Lambda' = F(\Lambda_1)$. Then $\mathrm{Span}_{\mathbb{R}}(\Lambda') = F(\mathrm{Span}_{\mathbb{R}}\{\Lambda_1\}) = F(V_1)$ is a complex subspace of V.

($\Leftarrow$) Given $\Lambda' \subset \Lambda$ such that $V_1 := \mathrm{Span}_{\mathbb{R}}\{\Lambda'\}$ is a complex subspace of V let $\Lambda_1 := \Lambda \cap V_1$ and
$F: V_1 \to V$ the inclusion and $f: Y := V_1/\Lambda_1 \to V/\Lambda$ the homomorphism induced by F. Then f is clearly an injective homomorphism. Λ_1 contains Λ' and thus has rank $\geq \mathrm{rk}_{\mathbb{Z}}(\Lambda') = \dim_{\mathbb{R}}(V_1) = 2\dim_{\mathbb{C}} V_1$. On the other hand Λ is a discrete subset of
1 $V \Rightarrow \Lambda_1 = V_1 \cap \Lambda$ is a discrete subset of V_1.

Hence, $rk(\Lambda_1) \le \dim_{\mathbb{C}}(V_1)$. Thus, $rk(\Lambda_1) = \dim_{\mathbb{C}}(V_1)$ and Y is a compact complex torus and f is an embedding of Y as a complex subtorus of X.

(b) Let Y be a subtorus of X such that the origin 0_X of X belongs to Y. Then Y is a subgroup of X. Indeed, if $f: Y \to X$ is the injective holo map and $h: Y \to X$ is the homomorphism $t_{-f(0_Y)} \circ f$ of ch 1 Prop 2.1 (a), and $0 = f(y_1)$, then

$\forall y \in Y$, $h(y) \overset{(*)}{=} f(y) - f(0_Y)$, so

$$h(y_1) = \underbrace{f(y_1)}_{0} - f(0_Y) \overset{(**)}{=} -f(0_Y).$$

So $\forall y \in Y$, $f(y) \overset{\text{by}(*)}{=} h(y) + f(0_Y) \overset{\text{by}(**)}{=} h(y) - h(y_1) =$

$$= h(y - y_1) \in \mathrm{Im}(h).$$

Thus, $Y = \mathrm{Im}(f) = \mathrm{Im}(h)$ is a subgroup, since h is a group homomorphism.

Part (a) shows that Y is determined by a sublattice Λ_1 of Λ. Hence, there are only a countable set of such subtori Y through 0_X.

(c) Let X be a 2-dim'l cpt cpx torus with period matrix $\begin{pmatrix} 1 & 0 & z_1 & z_3 \\ 0 & 1 & z_2 & z_4 \end{pmatrix}$,

such that $z_1, z_2, z_3, z_4 \in \mathbb{C}$, $\det\left(\mathrm{Im}\begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}\right) \neq 0$,

$z_1 \notin \mathbb{Q}$, $z_2 \notin \mathbb{Q}(z_1)$, $z_3 \notin \mathbb{Q}(z_1, z_2)$, and $z_4 \notin \mathbb{Q}(z_1, z_2, z_3)$.

One can choose, for example, $z_1 = \sqrt{-1}$, $z_2 = \sqrt{2}$, $z_3 = \sqrt{3}$, $z_4 = \sqrt{-5}$.

We claim that X does not contain any 1-dim'l cpt complex torus. It suffices to prove that $\Lambda = \mathrm{Span}_{\mathbb{Z}}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}z_1\\z_2\end{pmatrix}, \begin{pmatrix}z_3\\z_4\end{pmatrix}\right\}$

does not contain any rank 2 sublattice, which spans, over $\mathbb{R}$, a complex line in $\mathbb{C}^2$ (by part (a)).

We will need the following:

<u>Lemma:</u> Let K be a field contained in a field L. Assume that $z_1 \notin K$, $z_2 \notin K(z_1)$, and $\begin{pmatrix}0\\0\end{pmatrix} \neq \begin{pmatrix}t\\s\end{pmatrix} \in K^2$. Then $\det\begin{pmatrix} t & z_1 \\ s & z_2 \end{pmatrix} \neq 0$.

<u>Pf</u>: (By contradition).

Assume that $t z_2 - s z_1 = 0$. Then $t = 0$, since $z_2 \notin K(z_1)$. But then $z_1 = 0$ contradicting the assumption that $z_1 \notin K$. $\square$

Assume that $\lambda, \mu \in \Lambda$ are linearly indep over $\mathbb{Z}$ (so also over $\mathbb{Q}$). We prove that $\mathrm{span}_{\mathbb{R}}\{\lambda, \mu\}$ is NOT a complex line.

Assume otherwise. Then $\det\begin{pmatrix} \lambda & \mu \end{pmatrix} = 0$

Write $\lambda = a_1\begin{pmatrix}1\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\end{pmatrix} + a_3\begin{pmatrix}z_1\\z_2\end{pmatrix} + a_4\begin{pmatrix}z_3\\z_4\end{pmatrix}$

$$\mu = b_1\begin{pmatrix}1\\0\end{pmatrix} + b_2\begin{pmatrix}0\\1\end{pmatrix} + b_3\begin{pmatrix}z_1\\z_2\end{pmatrix} + b_4\begin{pmatrix}z_3\\z_4\end{pmatrix},$$

with $a_i, b_i \in \mathbb{Z}$.

$\mathbb{R}^2 = \mathrm{Span}_{\mathbb{R}}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is not a complex line so

one of a_3, a_4, b_3, b_4 is non-zero.

Case $a_4 = b_4 = 0$:

$$0 = \det\begin{pmatrix} a_1 + a_3 z_1 & b_1 + b_3 z_1 \\ a_2 + a_3 z_2 & b_2 + b_3 z_2 \end{pmatrix}$$

Interchanging the order of λ, μ, if necessary, and Adding a rational multiple of one of λ or μ to the other, W.M.A $a_3 = 0$, $b_3 \neq 0$. So

$$0 = \det\begin{pmatrix} a_1 & b_1 + b_3 z_1 \\ a_2 & b_2 + b_3 z_2 \end{pmatrix}.$$ But this contradicts the above Lemma.

We may thus assume that at least one of a_4 or b_4 is non-zero. Again, after possibly interchanging the roles of λ, μ, and after adding a rational multiple of one to the other, W.M.A $a_4 = 0$, $b_4 \neq 0$. Then

$$\det\begin{pmatrix} a_1 + a_3 z_1 & b_1 + b_3 z_1 + b_4 z_3 \\ a_2 \;\; a_3 z_2 & b_2 + b_3 z_2 + b_4 z_4 \end{pmatrix} = 0.$$

But this contradicts the above Lemma with $K = \mathbb{Q}(z_1, z_2)$. A contradition. We conclude that $\mathrm{span}_{\mathbb{R}}\{\lambda, \mu\}$ is not a complex line. $\square$

HW1 Prob 2 (B-Lange ch 1 prop. 2)

$X = V/\Lambda$

(a) Let $\mathcal{B} = \{\lambda_1, \dots, \lambda_{2g}\}$ be a basis of Λ and choose $\mathcal{E} = \{\underset{\overset{\shortparallel}{\lambda_{g+1}}}{e_1}, \dots, \underset{\overset{\shortparallel}{\lambda_{2g}}}{e_g}\}$ as a basis for V.

Then $\Pi = \left([\lambda_1]_{\mathcal{E}} \cdots [\lambda_g]_{\mathcal{E}} \; \underset{\overset{\shortparallel}{\begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}}}{[\lambda_{g+1}]_{\mathcal{E}}} \cdots \underset{\overset{\shortparallel}{\begin{pmatrix}0\\ \vdots\\0\\1\end{pmatrix}}}{[\lambda_{2g}]_{\mathcal{E}}} \right) = (Z \; I).$

(b) We know that

$$\det\begin{pmatrix} Z & I \\ \overline{Z} & I \end{pmatrix} \neq 0.$$

Now, using elementary column operations we get

$$\det\begin{pmatrix} Z & I \\ \overline{Z} & I \end{pmatrix} = \det\begin{pmatrix} i\,\mathrm{Im}(Z) & I \\ -i\,\mathrm{Im}(Z) & I \end{pmatrix}$$

and using elementary row operations we get

$$\det\begin{pmatrix} i\,\mathrm{Im}(Z) & I \\ -i\,\mathrm{Im}(Z) & I \end{pmatrix} = \det\begin{pmatrix} i\,\mathrm{Im}(Z) & I \\ 0 & 2I \end{pmatrix} = 2^g (i)^g \det(\mathrm{Im}(Z)).$$

In particular $\det(\mathrm{Im}(Z)) \neq 0$.

HW 1 Problem 3 (B-Lange Ch 1 prob 6)

(a) If J is a ~~two~~ complex structures on $\mathbb{R}^{2g}$ then $J^2 = -\underbrace{I}_{\text{the identity matrix}}$ and so, by a theorem in linear algebra, there exists a basis β of $\mathbb{R}^{2g}$ such that

$$[J]_\beta = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & & O \\ & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & \\ & & \ddots & \\ O & & & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix} =: J_0$$

So identifying J with its standard matrix, and letting $P = P_\beta$ be the change of basis matrix we get $J_0 = P^{-1} J P$, or equivalently, $J = P J_0 P^{-1}$.

Note that J_0 is the standard matrix of the standard complex structure on $\mathbb{C}^g$ with standard $\mathbb{R}$-basis $\{e_1, ie_1, e_2, ie_2, \ldots, e_g, ie_g\}$.

The map $GL_{2g}(\mathbb{R}) \to C_{2g}$:= set of complex str on $\mathbb{R}^{2g}$

$$P \longmapsto P J_0 P^{-1}$$

is thus surjective identifying C_{2g} with the conjugacy class of J_0.

The stabilizer of J_0 is $GL_g(\mathbb{C})$, hence we get the bijection

$$C_{2g} \xrightarrow{1:1} GL_{2g}(\mathbb{R}) / GL_g(\mathbb{C}).$$

(b) Let M_g be the set of isomorphism classes of g-dim'l cpt complex tori.
Let $\Lambda_0 := \mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$ be the standard lattice. Consider the map

$$\tilde{\kappa} : \mathcal{C}_{2g} \longrightarrow M_g$$

sending $J \in \mathcal{C}_{2g}$ to the isomorphism class of the compact complex torus $(\mathbb{R}^{2g}, J)/\Lambda_0$. Then $\tilde{\kappa}(J_1) = \tilde{\kappa}(J_2)$ if and only if there exists an isomorphism

$$P : (\mathbb{R}^{2g}, J_1) \xrightarrow{\simeq} (\mathbb{R}^{2g}, J_2)$$

(so that $J_2 P = P J_1 \iff J_2 = P J_1 P^{-1}$) and $P(\Lambda_0) = \Lambda_0$, by [B-Lange, Prop 2.1 (b)]. It follows that the fibers of $\tilde{\kappa}$ are the orbits in $\mathcal{C}_{2g}$ under the conjugation action by $J \mapsto PJP^{-1}$, where $P \in \{P \in GL_{2g}(\mathbb{R}) : P(\Lambda_0) = \Lambda_0\} = GL_{2g}(\mathbb{Z})$.

Thus $\tilde{\kappa}$ induces an injective map

$$GL_{2g}(\mathbb{Z}) \backslash GL_{2g}(\mathbb{R}) / GL_g(\mathbb{C}) \xrightarrow[\kappa]{} M_g.$$

It remains to prove that $\tilde{\kappa}$, and hence κ, is surjective. If $J = PJ_0P^{-1} \in \mathcal{C}_{2g}$, then P induces the isom $P : (\mathbb{R}^{2g}, J_0)/P^{-1}(\Lambda_0) \to (\mathbb{R}^{2g}, J)/\Lambda_0$.

As we vary P in $GL_{2g}(\mathbb{R})$, we realize every lattice in $\mathbb{R}^{2g}$ in the form $P^{-1}(\Lambda_0)$, hence $\tilde{\kappa}$ is surjective. $\square$

HW problem 4: X cpt cpx torus, $\mu: X\times X \to X$ the addition map $p_i: X\times X \to X$ the projection, $i=1,2$, ω a C^∞- 1-form on X. Show that ω is translation invariant $\Leftrightarrow \mu^*\omega = p_1^*\omega + p_2^*\omega$.

Proof:

$(\Leftarrow)$ Assume that $\mu^*\omega = p_1^*\omega + p_2^*\omega$. Let $e_{x_0}: X \to X\times X$ be $e_{x_0}(x) = (x_0, x)$. Let $t_{x_0}: X \to X$ be $t_{x_0}(x) = x + x_0$. Then $t_{x_0} = \mu \circ e_{x_0}$ and so

$$t_{x_0}^*(\omega) = e_{x_0}^*(\mu^*(\omega)) = e_{x_0}^*(p_1^*\omega + p_2^*\omega) =$$

$$\underbrace{(p_1\circ e_{x_0})}_{=0}{}^*(\omega) + \underbrace{(p_2\circ e_{x_0})}_{=id_X}{}^*(\omega) = 0 + id_X^*(\omega) = \omega.$$

$p_1\circ e_{x_0}$ is the constant map

We conclude that ω is translation invariant.

$(\Rightarrow)$ Let $(x_1, x_2) \in X\times X$.

Let $e_{x_1}: X \to X\times X$ be $e_{x_1}(x) = (x_1, x)$. Let $\ell_{x_2}: X \to X\times X$ be $\ell_{x_2}(x) = (x, x_2)$.

Assume that ω is translation invariant. Then

$\omega = t_{x_1}^*(\omega) = e_{x_1}^*(\mu^*(\omega))$ and, as we saw above,

$\omega = e_{x_1}^*(p_1^*(\omega) + p_2^*(\omega))$. So

(*) $\boxed{e_{x_1}^*(\mu^*(\omega) - p_1^*(\omega) - p_2^*(\omega)) = \omega - \omega = 0.}$

Similarly,

$\omega = t_{x_2}^*(\omega) = \ell_{x_2}^*(\mu^*(\omega))$ and

$\omega = \ell_{x_2}^*(p_1^*(\omega) + p_2^*(\omega))$. So

(**) $\boxed{\ell_{x_2}^*(\mu^*(\omega) - p_1^*(\omega) - p_2^*(\omega)) = 0.}$

The tangent space $T_{(x_1,x_2)}(X\times X)$ is the direct sum of the images of the differentials

$d_{x_2} e_{x_1} : T_{x_2} X \to T_{(x_1,x_2)}(X\times X)$ and

$d_{x_1} \beta_{x_2} : T_{x_1} X \to T_{(x_1,x_2)}(X\times X)$.

Hence, $(*) + (**) \Longrightarrow$ the 1-form $\mu^*(\omega)-p_1^*(\omega)-p_2^*(\omega)$ vanishes at $(x_1,x_2)\in X\times X$.

As (x_1,x_2) was an arbitrary point of $X\times X$ we get that $\mu^*(\omega)-p_1^*(\omega)-p_2^*(\omega)$ vanishes identically.

$\square$

HW Problem 5: Let Γ be a free abelian gp of even rank and set $V = \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$. Then Γ is a lattice in V and $X = V/\Gamma$ is a compact torus. Giving a complex structure on X is equivalent to giving a complex vector space structure on V (the universal cover).

Given a complex structure on V, let

$$J : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$$ be given by

$$J(v \otimes c) = iv \otimes c$$

and we regard $V \otimes_{\mathbb{R}} \mathbb{C}$ as a $\mathbb{C}$-vector space via the action of scalars on the second factor. Set

$H^{10} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ to be the i-eigenspace of J and

$H^{01} \subset V \otimes_{\mathbb{R}} \mathbb{C}$ " " $(-i)$-eigenspace of J.

Then

$$H^{10} = \{ v \otimes i + iv \otimes 1 : v \in V \}$$

Indeed,

$$J(v \otimes i + iv \otimes 1) = iv \otimes i + (i^2) v \otimes 1 = iv \otimes i - v \otimes 1$$
$$(v \otimes i + iv \otimes 1) i = v \otimes i^2 + iv \otimes i$$

Note that $iv \otimes i - v \otimes 1 = (iv) \otimes i + i(iv) \otimes 1$, so H^{10} is J invariant as well as a complex subspace.

$H^{01} = \{ -v \otimes i + iv \otimes 1 : v \in V \}$, since

$$J(-v \otimes i + iv \otimes 1) = -iv \otimes i + (i^2) v \otimes 1 = [(-iv \otimes 1) + (v \otimes i)] i$$
$$= (-v \otimes i + iv \otimes 1)(-i).$$

Clearly, $H^{01} = \overline{H^{10}}$, since conjugation is $\overline{v \otimes c} = v \otimes \bar{c}$.

Conversely, given a decomposition $V \otimes_{\mathbb{R}} \mathbb{C} = H^{10} \oplus H^{01}$ with $\overline{H^{10}} = H^{01}$, define $J : V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C}$ as acting by i on H^{10} and by $(-i)$ on H^{01}

Then

$$V = \text{subspace of } V \otimes_{\mathbb{R}} \mathbb{C} \text{ invariant under conjugation}$$

$$= \{ (\omega, \bar{\omega}) : \omega \in H^{10} \}.$$

So $J(\omega, \bar{\omega}) = (\omega i, \bar{\omega}(-i)) = (\omega i, \overline{\omega i}) \in V$

and $J^2 = -id_V$. Hence, J is a complex structure on V.

$\square$

HW 1 problem 6 (B-Lange Ch. 2 prob 11)

Our lattice $\Lambda \subset \mathbb{C}^g$ is the span of the columns of the period matrix $\Pi = (Z\ \ I)$.
Let λ_j be the j-th column of Π, $1 \le j \le 2g$ and set $\mathcal{B} = \{\lambda_1, \ldots, \lambda_{2g}\}$.
Let $\mathcal{E} := \{e_1 = \lambda_{g+1}, \ldots, e_g = \lambda_{2g}\}$ be the standard basis of $\mathbb{C}^g$.

We are given that the matrix

$$\left(E(\lambda_i, \lambda_j)\right) = \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \text{ is anti-symmetric,}$$

so $A^t = -A$ and $C^t = -C$.
We know that E belongs to $NS(X)$ if and only if $E(\lambda_k, \lambda_j) \in \mathbb{Z}$ and $E(i\lambda_k, i\lambda_j) = E(\lambda_k, \lambda_j)$ for all $1 \le k, j \le 2g$, by Prop. 1.6 in Ch. 2 of B-Lange.
Let $J : \mathbb{C}^g \to \mathbb{C}^g$ be the linear transformation of multiplication by i. Set

$$R := [J]_{\mathcal{B}}, \text{ the matrix of } J \text{ in the basis } \mathcal{B}.$$

We conclude that
(a) E belongs to $NS(X)$
$\Longleftrightarrow$
(a') A, B, C are integral and $\boxed{R^t \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} R \overset{(*)}{=} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}.}$

It remains to prove that ⊛ is equivalent to the equality

$$(**)\quad \boxed{A - BZ + Z^t B^t + Z^t C Z = 0.}$$

Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{C}^g \arrow[r, "J"] & \mathbb{C}^g \\
\mathbb{C}^g \arrow[u, "[\,]_{\mathcal{E}}"] \arrow[d, "[\,]_{\mathcal{B}}"'] \arrow[r, "J"] & \mathbb{C}^g \arrow[u, "[\,]_{\mathcal{E}}"'] \arrow[d, "[\,]_{\mathcal{B}}"] \\
\mathbb{R}^{2g} \arrow[r, "\text{mult by } R"'] \arrow[uu, bend left=60, "\text{mult by } \Pi"] & \mathbb{R}^{2g} \arrow[uu, bend right=60, "\text{mult by } \Pi"']
\end{tikzcd}$$

We get that $i\Pi = \Pi R$. Hence

$$\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} \underbrace{\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}}_{\begin{pmatrix} Z & I \\ \overline{Z} & I \end{pmatrix}} = \underbrace{\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}}_{\begin{pmatrix} Z & I \\ \overline{Z} & I \end{pmatrix}} R$$

Invertible, by problem 2

$\Longleftrightarrow$

$$\begin{pmatrix} Z & I \\ \overline{Z} & I \end{pmatrix}^{-1} \begin{pmatrix} iZ & iI \\ -i\overline{Z} & -iI \end{pmatrix} = R.$$

Write $\boxed{Z = X + iY}$.

Then $$\begin{pmatrix} Z & I \\ \overline{Z} & I \end{pmatrix}^{-1} = \frac{1}{2i} \begin{pmatrix} Y^{-1} & -Y^{-1} \\ (-XY^{-1} + iI) & iI + XY^{-1} \end{pmatrix}.$$

We get that

$$R = \begin{pmatrix} Y^{-1}X & Y^{-1} \\ -[Y + XY^{-1}X] & -XY^{-1} \end{pmatrix},$$ by a direct computation.

Set $M = \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$.

Now $R^{-1} = [\mathcal{J}]_{\mathcal{B}}^{-1} = [\mathcal{J}^{-1}]_{\mathcal{B}} = [-\mathcal{J}]_{\mathcal{B}} = -R$.

Hence, equation ⊛ is equivalent to $MR = -R^t M = (MR)^t$.

So ⊛ $\iff$ $\underbrace{\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} Y^{-1}X & Y^{-1} \\ -[Y + XY^{-1}X] & -XY^{-1} \end{pmatrix}}_{}$ is symmetric.

$$\begin{pmatrix} AY^{-1}X - BY - BXY^{-1}X & AY^{-1} - BXY^{-1} \\ -[B^tY^{-1}X + CY + CXY^{-1}X] & -B^tY^{-1} - CXY^{-1} \end{pmatrix}$$

So ⊛ is equivalent to

(1) $AY^{-1}X - BY - BXY^{-1}X$ is symmetric

(2) $-B^tY^{-1} - CXY^{-1}$ is symmetric

(3) $AY^{-1} - BXY^{-1} = -[B^tY^{-1}X + CY + CXY^{-1}X]^t$.

Now (2) $\iff -B^tY^{-1} - CXY^{-1} + (Y^t)^{-1}B + (Y^t)^{-1}X^tC^t = 0$

$\iff$

(2') $\boxed{-Y^tB^t - Y^tCX + BY - X^tCY = 0.}$

(3) $\iff AY^{-1} - BXY^{-1} + X^t(Y^{-1})^tB + Y^tC^t + X^t(Y^{-1})^tX^tC^t = 0$

$\iff A - BX + X^t(Y^{-1})^t\underbrace{[BY - X^tCY]}_{\overset{(2')}{=}\, Y^tB^t + Y^tCX} - Y^tCY = 0$

(3') $\iff \boxed{A - BX + X^tB^t + X^tCX - 3\,Y^tCY = 0}$

Equations (2′) and (3′) are the imaginary and real parts of the equation

$$\boxed{(+) \quad A - BZ + Z^t B^t + Z^t C Z = 0.}$$

It remains to prove that equation (1) follows from (2′) and (3′).

$(1) \Leftrightarrow (A - BX)(Y^{-1}X) - Y^t CX$ is symmetric

$\underset{\text{by (3')}}{\Leftrightarrow}$ $-X^t B^t Y^{-1} X - X^t C X Y^{-1} X$ is symmetric

$\Leftrightarrow$

$0 = -X^t B^t Y^{-1} X - X^t C X Y^{-1} X + X^t (Y^{-1})^t B X + X^t (Y^{-1})^t X^t C^t X$

$\Leftrightarrow$

$$0 = X^t (Y^{-1})^t \underbrace{\left[-Y^t B^t - Y^t C X + B Y - X^t C Y \right]}_{\overset{\text{by (2')}}{=} 0} (Y^{-1} X)$$

So indeed (2′) and (3′) $\Rightarrow$ (1). $\square$

HW 1 Problem 7: Need to show: The rank $\rho(X)$ of $NS(X)$ is $\leq h^{11}(X)$.

Proof: By defintion, $NS(X) := \mathrm{Im}\left(c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})\right)$ is a subgroup of $H^2(X, \mathbb{Z})$. We have $H^2(X, \mathbb{C}) = H^2(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, by the Universal Coefficient Theorem. Hence, $\mathrm{Span}_{\mathbb{C}}\{NS(X)\}$ is a complex subspace of $H^2(X, \mathbb{C})$ of dimension $\rho(X)$. Proposition 1.6 in Ch 2 of B-Lange shows that $NS(X)$ is contained in $H^{1,1}(X) \subset$ (the complex subspace of $H^2(X, \mathbb{C})$).
Hence, $\rho(X) := rk(NS(X)) = \dim(\mathrm{Span}_{\mathbb{C}}(NS(X))) \leq \dim H^{1,1}(X) = h^{1,1}(X)$.

$\square$