HW4 Q1:

(Exercise 2 in Ch 2 of Birkenhake-Lange)

Part a: $X = V/\Lambda$, $L = L(H,\chi)$, $\{e_1, \dots, e_g\}$ a basis for V and $v_1, \dots, v_g$ the corresponding coordinates on V. Then

$$c_1(L) = \frac{i}{2} \sum_{j=1}^{g} \sum_{k=1}^{g} H(e_j, e_k)\, dv_j \wedge d\bar{v}_k.$$

$\pi: V \to V/\Lambda = X$.

Proof: The canonical factor of L is

$$a_L(\lambda) = \chi(\lambda)\, e^{\pi H(v,\lambda) + \frac{\pi}{2} H(\lambda,\lambda)}.$$

Let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of X by simply connected open subsets, such that the index α of U_α determines also a connected component $\tilde{U}_\alpha$ of $\pi^{-1}(U_\alpha) \subset V$. Then the Čech cocycle for L in $Z^1(\{U_\alpha\}, \mathcal{O}_X^*)$ is given by

(as we saw) $\{f_{\mu\nu}\}$, $f_{\mu\nu} \in H^0(U_\mu \cap U_\nu, \mathcal{O}_X^*)$, where

$$f_{\mu\nu} \circ \underbrace{\pi_\nu}_{\pi|\tilde{U}_\nu} = \chi(\lambda_{\mu\nu})\, e^{\pi H(v,\lambda_{\mu\nu}) + \frac{\pi}{2} H(\lambda_{\mu\nu},\lambda_{\mu\nu})},$$

where $\lambda_{\mu\nu} \in \Lambda$ is the unique element such that $(\tilde{U}_\mu + \lambda_{\mu\nu}) \cap \tilde{U}_\nu \neq \emptyset$. We compute the image in $H^2(X,\mathbb{Z}) \subset H^2_{DR}(X,\mathbb{R})$ of $[\{f_{\mu\nu}\}] \in H^1(X, \mathcal{O}_X^*)$

via the hint given in HW4 Q1,

$$\frac{1}{2\pi i}\, d\log(f_{\mu\nu}) = \frac{1}{2\pi i} \sum_{j=1}^{g} \left(\frac{\partial f_{\mu\nu}}{\partial v_j}\right)\Big/ f_{\mu\nu}\; dv_j =$$

$$= \sum_{j=1}^{g} \frac{\pi}{2\pi i} \left[\frac{e^{\pi H(v,\lambda_{\mu\nu})} \cdot H(e_j, \lambda_{\mu\nu})}{e^{\pi H(v,\lambda_{\mu\nu})}} \right] dv_j = \frac{1}{2i} \sum_{j=1}^{g} H(e_j, \lambda_{\mu\nu})\, dv_j$$

$$= \underbrace{\frac{1}{2i}\sum_{j=1}^{g} H(e_j, v+\lambda_{\mu\nu})\,dv_j}_{\pi_\nu^*(\varphi_\nu)} - \underbrace{\frac{1}{2i}\sum_{j=1}^{g} H(e_j, v)\,dv_j}_{\pi_\mu^*(\varphi_\mu)}.$$

Define $\{\varphi_\mu\}$ by

Note that $\pi_\mu = \pi_\nu \circ t_{\lambda_{\mu\nu}}$ on $(\tilde{U}_\mu + \lambda_{\mu\nu}) \cap \tilde{U}_\nu$.

We see that $\frac{1}{2\pi i} d\log(f_{\mu\nu})$ is the Čech coboundary of the 0-th Čech co-chain of 1-forms $\{\varphi_\mu\}_{\mu\in I}$. Now $dd\log(f_{\mu\nu}) = 0$.

Thus the 0-cochain of 2-forms $\{d\varphi_\mu\}_{\mu\in I}$ glues to a global 2-form.

The latter represents $c_1(L)$ by the Hint.

Reason for the Hint:

For any local choice of logs on U_α, the Čech codifferential δ yield a Čech 2-cocycle

$$\delta(\log f_{\mu\nu}) \in Z^2(\{U_\alpha\}, \mathbb{Z}) \subset Z^2(\{U_\alpha\}, \mathbb{C}).$$

Now use the Čech-de Rham Thm, and its proof via the exact sequences of sheaves

$$0 \to \mathbb{C} \to A^0 \to Z_d^1 \to 0$$

$$0 \to Z_d^1 \to A^1 \to Z_d^2 \to 0$$

(A^i sheaf of C^∞-i-forms, Z_d^i sheaf of closed i-forms),

to map $\delta(\log f_{\mu\nu})$ via the isom

$$\check{H}^2(\{U_\alpha\}, \mathbb{C}) \simeq \check{H}^1(\{U_\alpha\}, Z_d^1) \simeq \check{H}^0(\{U_\alpha\}, Z_d^2)$$

to the global 2-form as we did above.

Calculating the global 2-form:

$$d\varphi_\mu = \frac{1}{2i} \sum_{j=1}^{g} \left(\sum_{k=1}^{g} \frac{\partial H(e_j, v)}{\partial \bar{v}_k} \, d\bar{v}_k \right) \wedge dv_j$$

Use the fact that $H(e_j, v)$ is the complex conjugate of the linear, hence holomorphic, function $H(v, e_j)$ in v. So $dH(e_j, v) = \bar{\partial} H(e_j, v)$, since $\partial H(e_j, v) = 0$.

$$= \underbrace{-\frac{1}{2i}}_{i/2} \sum_{j=1}^{g} \sum_{k=1}^{g} H(e_j, e_k) \, dv_j \wedge d\bar{v}_k .$$

This completes the proof of Part a.

Part (b): Let $\{x_1, \dots, x_g, y_1, \dots, y_g\}$ be the real coordinates functions on V with respect to a symplectic basis of Λ for L. Then

$$c_1(L) = -\sum_{j=1}^{g} d_j \, dx_j \wedge dy_j$$

where $(d_1, \dots, d_g)$ is the type of L.

Proof: Let $A^{p,q}$ be the sheaf of C^∞-forms of type (p,q) on X and on V. The operators $\bar{\partial}: A^0 \to A^{0,1}$ and $\partial: A^{0,1} \to A^{1,1}$ are independent of the coordinates, and depend only on the complex structure. Part (a) shows that $c_1(L)$ corresponds to the translation invariant $(1,1)$-form $(i/2)\,\partial\bar{\partial} f$, where $f \in A^0(V)$ is the function $H(v, v)$. We need to show that

$$\left(\frac{i}{2} \partial \bar{\partial} f\right)\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right) \underset{(3)}{=} -\delta_{jk} \, d_j = -\operatorname{Im} H\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_k}\right)$$

where we identify translation invariant vector fields on V with elements of V.
(So $\frac{\partial}{\partial x_j}$ is identified with λ_j, $\frac{\partial}{\partial y_K}$ with μ_K etc.)

Eliminating coordinates, we need to verify the equality

$$\left(\frac{i}{2}\partial\bar\partial f\right)(\xi_1, \xi_2) = -\overbrace{\mathrm{Im} H(\xi_1, \xi_2)}^{\text{a constant function on } V}$$

for every two translation invariant vector fields ξ_1, ξ_2 on V. It suffices to verify the equality for one basis of V as an $\mathbb{R}$-vector space.

Choose a basis $\{e_1, \dots, e_g\}$ of V over $\mathbb{C}$ which is orthogonal w.r.t. H, $H(e_j, e_K) = 0$, if $j \neq K$. Let $z_1 = \tilde{x}_1 + i\tilde{y}_1, \dots, z_g = \tilde{x}_g + i\tilde{y}_g$ be the associated complex coordinates. Then e_j corresponds to $\frac{\partial}{\partial \tilde{x}_j}$ and ie_j to $J\left(\frac{\partial}{\partial \tilde{x}_j}\right) = \frac{\partial}{\partial \tilde{y}_j}$, where J is the complex structure of V.

$$\frac{i}{2}\partial\bar\partial f \underset{\text{Part (a)}}{=} \frac{i}{2}\sum_{j=1}^{g} H(e_j, e_j)\, \underbrace{dz_j}_{d\tilde{x}_j + i d\tilde{y}_j} \wedge \underbrace{d\bar{z}_j}_{d\tilde{x}_j - i d\tilde{y}_j} =$$

$$= \frac{i}{2}\sum_{j=1}^{g} H(e_j, e_j)(-2i)\, d\tilde{x}_j \wedge d\tilde{y}_j = \sum_{j=1}^{g} H(e_j, e_j)\, d\tilde{x}_j \wedge d\tilde{y}_j$$

So $\left(\frac{i}{2}\partial\bar\partial f\right)\left(\frac{\partial}{\partial \tilde{x}_j}, \frac{\partial}{\partial \tilde{y}_K}\right) = H(e_j, e_K) = H\left(\frac{\partial}{\partial \tilde{x}_j}, \frac{\partial}{\partial \tilde{x}_K}\right)$.

Now $\frac{\partial}{\partial \tilde{y}_K} = \underset{\text{The complex structure}}{J}\left(\frac{\partial}{\partial \tilde{x}_K}\right)$ and $-\mathrm{Im} H\left(\frac{\partial}{\partial \tilde{x}_j}, J\left(\frac{\partial}{\partial \tilde{x}_K}\right)\right) = +\mathrm{Im} H\left(J\left(\frac{\partial}{\partial \tilde{x}_K}\right), \frac{\partial}{\partial \tilde{x}_j}\right)$

$= +H\left(\frac{\partial}{\partial \tilde{x}_K}, \frac{\partial}{\partial \tilde{x}_j}\right)$ as claimed. $\square$

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