Associated to a compact Riemann surface X and a very ample divisor $D \in Div(X)$, we get a holomorphic embedding $\phi_D : X \to \mathbb{P}L(D)^*$. The following sequence of exercises indicates how one can obtain equations for $\phi_D(X)$ in the projective space $\mathbb{P}L(D)^*$. The general discussion requires knowledge of the dimension of the vector spaces $L(dD)$, $d \geq 1$, provided by Riemann-Roch Theorem. We restrict ourselves to the genus 0 and 1 case, where we already know these dimensions.

1. Let V be an $n+1$-dimensional vector space. Show that the vector space $\mathrm{Sym}^d(V^*)$, of homogeneous polynomial functions on V of degree $d \geq 1$, has dimension $\binom{n+d}{d}$. Note, that if $\{v_0, v_1, \ldots, v_n\}$ is a basis of V, and $\{x_0, x_1, \ldots, x_n\}$ is the dual basis of V^*, then $\mathrm{Sym}^d(V^*)$ is precisely the space of homogeneous polynomials of degree d in the x_i's.

2. Let X be a compact Riemann surface, $D \in \mathrm{Div}(X)$, with a base-point-free linear system $|D|$, and $\phi_D : X \to |D|^* \cong \mathbb{P}L(D)^*$ the associated holomorphic map. Let d be a positive integer, and consider the multiplication map (induced by multiplication in the field $\mathcal{M}(X)$ of meromorphic functions)
$$\mu_d \ : \ \mathrm{Sym}^d L(D) \quad \longrightarrow \quad L(dD) \tag{1}$$
   (a) Show that a degree d homogeneous polynomial $G \in \mathrm{Sym}^d L(D)$ vanishes identically on the image $\phi_D(X) \subset \mathbb{P}L(D)^*$, if and only if $\mu_d(G) = 0$. If $\mu_d(G) \neq 0$, show that
$$\mathrm{div}(\mu(G)) + dD \quad = \quad \phi_D^*(G),$$
   where the divisor $\phi_D^*(G)$ is defined in the paragraph following definition V.4.12 page 159 in the text.

   (b) Let $\mathcal{P} := \oplus_{d=0}^{\infty} \mathrm{Sym}^d L(D)$ be the homogeneous coordinate ring of $\mathbb{P}L(D)^*$ (the polynomial algebra). The homomorphisms μ_d fit into a graded $\mathbb{C}$-algebra homomorphism
$$\mu \ : \ \mathcal{P} \quad \longrightarrow \quad \oplus_{d=0}^{\infty} L(dD),$$
   where the multiplication on the right hand side is induced by multiplication in $\mathcal{M}(X)$. Show that $\ker(\mu)$ is a prime ideal.

Definition: Let X be a compact Riemann surface, $D \in \mathrm{Div}(X)$ a very ample divisor (see Proposition V.4.20 page 163), and $\phi_D : X \to |D|^*$ the corresponding holomorphic embedding. The ring $\mathcal{P}/\ker(\mu)$ is called the *homogeneous coordinate ring of* $\phi_D(X)$. The embedding ϕ_D is said to be *projectively normal*, if the homomorphisms μ_d are surjective, for all $d \geq 0$.

3. Set $\infty := [0:1] \in \mathbb{P}^1$, $D = n \cdot \infty \in \mathrm{Div}(\mathbb{P}^1)$, $n \geq 1$.

   (a) Show that the homomorphism μ_d, given in (1), is surjective, for all $d \geq 1$. Conclude, that the rational normal curve of degree n in $\mathbb{P}^n$ is projectively normal.

   (b) Set $n := 3$, $d = 2$. Show that the subspace $\ker(\mu_2)$ of $\mathrm{Sym}^2 L(3\infty)$, of quadrics vanishing identically on the twisted cubic curve, is three-dimensional. Compare with the basis for $\ker(\mu_2)$ given in problem V.2.F page 145.

4. Let $X := \mathbb{C}/L$ be a compact torus, $p_0 := 0 + L \in X$ the origin, $D := np_0 \in \mathrm{Div}(X)$.

   (a) Show that μ_2, given in (1), is not surjective, if $n = 2$.

   (b) Show that for $n \geq 2$, a subset $\{f_0, f_2, f_3, \ldots, f_n\}$ of $L(np_0)$, satisfying $\mathrm{ord}_{p_0}(f_i) = -i$, is a basis for $L(np_0)$. Furthermore, such a basis exists.

(c) Assume $n \geq 3$. We know that D is very ample. Show that the homomorphism μ_d, given in (1), is surjective, for all $d \geq 1$. Conclude, that the elliptic normal curve of degree n in $\mathbb{P}^{n-1}$ is projectively normal. Hint: Prove the equality of sets,

$$\{0, 2, 3, \ldots, k\} + \{0, 2, 3, \ldots, m-1, m\} \quad = \quad \{0, 2, 3, \ldots, k+m-1, k+m\}$$

for all integers $k \geq 2$ and $m \geq 3$. Use it to prove that the multiplication homomorphism

$$L(kp_0) \otimes L(mp_0) \quad \longrightarrow \quad L((k+m)p_0)$$

is surjective.

(d) Let $C \subset \mathbb{P}^3$ be the image of X via the embedding ϕ_{4p_0}. Show that the vector space $\ker(\mu_2)$ of quadrics (homogeneous polynomials $Q(x_0, x_1, x_2, x_3)$ of degree 2), vanishing identically on C, is two dimensional. Use problem 2b to show that any non-zero quadric in $\ker(\mu_2)$ is irreducible. Let $\{Q_1, Q_2\}$ be a basis for $\ker(\mu_2)$ and assume, that the complete intersection $Q_1 \cap Q_2$, of the two quadric surfaces $Q_i = 0$, is smooth and connected. Prove that $Q_1 \cap Q_2$ is equal to C. (C is always the complete intersection $Q_1 \cap Q_2$, by a version of Bezout's Theorem, so the smoothness and connectedness assumptions are not needed). Compare with problem I.3.F page 18.