Let g be a non-negative integer, $\{\lambda_1, \lambda_2, \ldots, \lambda_{2g+1}\}$ distinct complex numbers, and X_0 the affine algebraic curve in $\mathbb{C}^2$ given by $y^2 - \prod_{i=1}^{2g+1}(x-\lambda_i) \quad = \quad 0.$

Let X be the compact hyperelliptic Riemann surface, containing X_0, which is a branched double cover $\pi : X \to \mathbb{P}^1$, whose restriction to X_0 is equal to the restriction of the function x from $\mathbb{C}^2$ to X_0. Then π is branched over $\{\lambda_1, \lambda_2, \ldots, \lambda_{2g+1}, \infty\}$ (Miranda, Lemma III.1.7, page 60).

1. Denote the restriction of x and y to X_0 by x and y as well, and regard them as meromorphic functions on X. Determine the zeroes and poles, and their multiplicities, for the following meromorphic one forms: i) $\frac{dy}{x}$, ii) $\frac{dx}{y}$ (you should get that the latter is holomorphic, for $g \geq 1$). For each of the above forms show that the difference, between the number of zeroes and the number of poles, counted with multiplicities, is equal to $2g-2$.

2. Let ω be a holomorphic 1-form on X satisfying $\iota^*(w) = \omega$, where ι is the Hyperelliptic involution (Miranda, Lemma III.1.9 page 61). Prove that $\omega = 0$.

3. Prove that every holomorphic 1-form on X is of the form $f(x)dx/y$, where f is a polynomial of degree $\leq g-1$. Conclude that the complex vector space of global holomorphic 1-forms on X is g-dimensional.