Riemann Surfaces Homework Assignment 3, Fall 2004

Due Thursday, September 28.

1. For which values of $\lambda \in \mathbb{C}$ is the projective cubic curve C_λ
$$x^3 + y^3 + z^3 + 3\lambda xyz = 0$$
in $\mathbb{P}^2$ singular? Show that each singular curve C_λ in the family is a union of three lines.

2. For which values of $\lambda \in \mathbb{C}$ is the projective cubic curve C_λ
$$y^2z - x(x - z)(x - \lambda z) = 0$$
in $\mathbb{P}^2$ singular? Show that all curves in the family are irreducible. Show that if C_λ is singular, than it has a single singular point p_0, and $C_\lambda \setminus \{p_0\}$ is isomorphic to $\mathbb{C} \setminus \{0\}$.

3. Show that the projective cubic plane curve C, given by $zy^2 - x^3$, is irreducible and has precisely one singular point p_0. Show that $C \setminus \{p_0\}$ is isomorphic to $\mathbb{C}$.

4. Miranda, section II.2 page 38: B, F.

5. Let w_1, w_2, be two complex numbers, which are linearly independent over $\mathbb{R}$, and let L be the lattice $\mathbb{Z}w_1 + \mathbb{Z}w_2$. The Weierstrass $\mathcal{P}$-function is
$$\mathcal{P}(z) \quad := \quad \frac{1}{z^2} + \sum_{w \in L, w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right).$$
It is shown in standard complex analysis textbooks (Lang, Ahlfors, etc ...) that $\mathcal{P}$ is absolutely convergent uniformly on compact subsets of $\mathbb{C} \setminus L$ and $\mathcal{P}$ is L-periodic, and so descends to a meromorphic function on the complex torus $X := \mathbb{C}/L$ (L-periodicity is obvious for the derivative $\mathcal{P}'$). Note, that $\mathcal{P}$ is even $\mathcal{P}(z) = \mathcal{P}(-z)$. It is characterized as the unique L-periodic function, holomorphic on $\mathbb{C} \setminus L$, whose Laurent series centered at 0,
$$\sum_{n=-\infty}^{\infty} a_n z^n$$
satisfies $a_{-2} = 1$, and $a_n = 0$, for all $n \leq 0$, except a_{-2}.

   (a) Show that the derivative $\mathcal{P}'$ has a simple zero at each half period
   $$\{u \ : \ u \in (\mathbb{C} \setminus L) \text{ and } 2u \in L\}$$
   (corresponding to three points in X). *Hint:* Use the fact that $\mathcal{P}$ is even and Lemma 3.14 in the text.

   (b) Assume now that $w_1 = 1$, $w_2 = \tau$, and $u \in \{\frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}\}$. Let $\theta^{(u)}(z)$ be the function defined in page 34 of the text. Show that the difference $\mathcal{P}(z) - \mathcal{P}(u)$ is a constant multiple of
   $$\frac{\theta^{(u)}(z)\theta^{(-u)}(z)}{[\theta^{(0)}(z)]^2}.$$