Let $F_i(x_0, x_1, \ldots, x_n)$ be homogeneous polynomials, $1 \leq i \leq n-1$, with $\deg(F_i) = d_i \geq 1$. Assume that their complete intersection X, given by $F_1 = F_2 = \ldots = F_{n-1} = 0$, is a smooth connected projective curve, so that X is a Riemann surface holomorphically embedded in $\mathbb{P}^n$. You may assume the following generalization of Proposition V.2.12 page 143 in the text:

Theorem: (A special case of Bezout's Theorem) The degree of X in $\mathbb{P}^n$, in the sense of Definition V.2.11 page 142 in the text, is the product

$$\deg(X) = \prod_{i=1}^{n-1} d_i.$$

1. Assume, without loss of generality, that X is disjoint from the linear subspace $x_0 = x_n = 0$, and let

$$\pi : X \longrightarrow \mathbb{P}^1$$

be the holomorphic map given by $[x_0 : x_1 : \cdots : x_n] \mapsto [x_0 : x_n]$. Assume, without loss of generality, that π is not a constant map. Prove the following generalization of Lemma V.2.14 page 143 in the text.

Lemma: The ramification divisor R_π of π is equal to $\mathrm{div}(\det(A))$, where A is the $(n-1) \times (n-1)$ polynomial matrix, whose (i, j) entry is $\frac{\partial F_i}{\partial x_j}$, $1 \leq i, j \leq n-1$.

Hint: Fix $p \in X$. We need to prove the equality

$$R_\pi(p) = \mathrm{div}(\det(A))(p).$$

Localize to one of the two affine chart $x_0 = 1$ or $x_n = 1$ (say $x_0 = 1$). Use first the Implicit Function Theorem to prove the equality of the supports of the two divisors. Assume next that p belongs to the support of R_π. Choose $0 < i < n$, so that $y_i := x_i/x_0$ is a local coordinate around p. Say $i = 1$. Use the Implicit Function Theorem to express the derivatives $\frac{\partial y_j}{\partial y_1}$ as solutions of a system of linear equations with the partials of the equations $F(1, y_1, \ldots, y_n)$ as coefficients. Now use Cramer's Rule to relate $\mathrm{ord}_p\left(\frac{\partial y_n}{\partial y_1}\right)$ to $\mathrm{div}(\det(A))(p)$.

2. Use Bezout's Theorem to prove that the genus g of X is given by the formula

$$2g - 2 = \left[\left(\sum_{i=1}^{n-1} d_i\right) - n - 1\right] \prod_{i=1}^{n-1} d_i.$$

3. Use the above to answer again Problem I.3.F page 18.