Kirwan's book section 2.5 page 46 Problem 2.10: A hint.

Let $N \subset \mathbb{P}^2$ be a subset consisting of nine distinct point satisfying the following two conditions:

(a) N is not contained in a single line.

(b) Given $P, Q \in N$, $P \neq Q$, the line $\overline{PQ}$ through P and Q contains a third point $R \in N \setminus \{P, Q\}$.

Part 1: Prove, that there exists a unique such configuration N of points, up to a projective linear transformations of $\mathbb{P}^2$.

Existence: The existence part is proven by checking, that the explicit set of nine points given in the problem

$$S \quad := \quad \{[0,1,-1],[0,1,\alpha],\ldots\},$$

where α is a primitive sixth root of unity, satisfies conditions (a) and (b). Following is an elegant way of organizing the check. Consider the group $G \subset PGL(3,\mathbb{C})$, of projective linear transformations of $\mathbb{P}^2$, generated by the following two elements $\{\sigma, \tau\}$:

$$\begin{aligned} \sigma(x,y,z) &= (y,z,x), \\ \tau(x,y,z) &= (x,\alpha^2 y,\alpha^4 z). \end{aligned}$$

Note, that both σ and τ have order 3 and they commute:

$$\sigma\tau[x,y,z] = [\alpha^2 y, \alpha^4 z, x] = [y, \alpha^2 z, \alpha^4 x] = \tau\sigma[x,y,z].$$

Thus, G is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Claim 1 *S is the G-orbit of the point $P := [0,1,-1]$*

$$S \quad = \quad \{g(P) \ : \ g \in G\}. \tag{1}$$

Recall, that in $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ there are precisely four cyclic subgroups of order 3.

Claim 2 *Given a point $Q \in S$, the four subsets of S, each consisting of three co-linear points (on the same line) one of which is Q, are in one-to-one correspondence with the four cyclic subgroups of G of order 3 (each "line" being the orbit of Q under one of the four subgroups).*

Do not check all 12 lines! Find a more efficient argument (checking 4 lines or less).

Remark 3 (This remark is not essential for the solution of the problem) The group G may be considered as the two dimensional vector space over the field of three elements. Then the 12 lines in the corresponding affine plane are: 4 lines through the origin, each a cyclic subgroup of G. Each line through the origin has two additional cosets resulting in 8 additional lines. The group of symmetries of the affine plane of G is the group $\mathrm{Aff}(G)$ of affine automorphisms, generated by the normal subgroup of translations and the subgroup $GL(2,\mathbb{Z}/3\mathbb{Z})$ of linear transformations. Claims 1 and 2 identify S with the affine plane of G. Combined with Lemma 2.22 in Kirwan's book one concludes: *The maximal subgroup of $PGL(3,\mathbb{C})$, which leaves the subset S invariant, is isomorphic to the group* $\mathrm{Aff}(G)$.

Uniqueness: We need to find an element of $PGL(3, \mathbb{C})$, which maps N to S. Motivated by Claim 1, it is tempting to introduce a group structure on N, depending on the choice of some point $P \in N$. Let us do that, except that we will not prove associativity. The associativity will follow from the existence and uniqueness.

Claim 4 *Each line in $\mathbb{P}^2$ intersects N along at most 3 points.*

Using Claim 4, we define a commutative binary operation. (i) We define P to be the identity $P + Q = Q$. (ii) The sum of three co-linear points is P. (iii) Every point is of order dividing 3; i.e., $3Q = P$. It follows, that $Q + Q = -Q$ is the third point of N on the line $\overline{PQ}$. Given $Q, R \in N \setminus \{P\}$, the third point of N on the line $\overline{QR}$ is $-(Q + R)$.

Claim 5 *Let $Q, R \in N \setminus \{P\}$, such that P, Q, and R are not co-linear. Then Q and R generate N. In other words, any one of the other six points can be expressed as a "word" of the form $P_1 + (P_2 + (\ldots)))$, where P_i is either Q or R.*

Uniqueness now follows from Lemma 2.22 in Kirwan's book.

Part 2: Prove that a projective curve of degree 3 passes through all the points of S, if and only if it is defined by a polynomial of the form

$$t(x^3 + y^3 + z^3) + 3\lambda xyz, \tag{2}$$

for some $[t, \lambda] \in \mathbb{P}^1$.

An elegant proof uses the G-action of Claim 1 on the 10-dimensional vector space V of homogeneous cubic polynomials in $\mathbb{C}[x, y, z]$. The subspace V_S, of polynomials vanishing along the points of S, is G-invariant. V decomposes as a direct sum of a 2-dimensional subspace V^G of invariant polynomials, given in equation (2), and 8 one-dimensional subspaces V_χ, corresponding to the 8 non-trivial characters of G (i.e., the non-trivial homomorphisms $\chi : G \to \mathbb{C}^*$ into the multiplicative group of non-zero complex numbers). One easily checks, that the subspace V^G is contained in V_S. Hence, $V_S = V^G \oplus W$, where W must be a direct sum of some of the V_χ. Show, that $W = (0)$, by showing that a non-zero polynomial in V_χ does not vanish along any of the points of S. A direct check is easy but a little tedious.

Remark 6 There is a more conceptual proof of the equality $W = (0)$, which uses Remark 3. Consider the 8-dimensional vector space $E := \mathbb{C}^G/\mathbb{C}$, consisting of maps from G to $\mathbb{C}$ modulo the subspace of constant functions. Then E is a representation of the group $\mathrm{Aff}(G)$ and E does not have any $\mathrm{Aff}(G)$-invariant proper subspace. Now $\mathbb{P}(V/V_S)$ embeds in $\mathbb{P}(E)$ as an $\mathrm{Aff}(G)$-invariant projective linear subspace. Hence, either V/V_S is 8-dimensional, which means that V_S is 2-dimensional and hence $W = (0)$, or $V = V_S$. The latter is clearly false, as there exist cubic curves not containing S.

Part 3: A curve defined by a polynomial of the form (2) is singular precisely for $[t, \lambda] \in \{[0, 1], [1, -1], [1, \alpha], [1, \bar{\alpha}]\}$, in which case it is a union of three lines.

The proof is easy.

Remark 7 Problem 2.10 is related to the fact, that a smooth complex cubic curve Σ in $\mathbb{P}^2$ is homeomorphic to the torus $S^1 \times S^1$. The circle S^1 has a natural group structure (multiplication of complex numbers of absolute value 1). The homeomorphism $\Sigma \cong S^1 \times S^1$ endows Σ with a group structure. This group structure can be defined geometrically in terms of co-linear points as above (section 3.2 Theorem 3.28). The nine points of the subset S in equation (1) are then the subgroup of points of order three on a smooth Σ containing S.