Riemann Surfaces Homework Assignment 8 Fall 2004

**Optional problems on line bundles.**

The significance of Forster's Problem 16.4 page 131 is explained in problems 1 and 2 below.

**Definition:**

1. Let $X$ be a topological space. A rank $n$ *complex vector bundle* over $X$ is a topological space $E$ with a continuous map $\pi : E \to X$, such that each fiber $\pi^{-1}(x)$, $x \in X$, is provided with the structure of a complex vector space, and $\pi : E \to X$ is locally trivial in the following sense:

   (a) For each $x \in X$, there is an open neighborhood $U$ of $x$ and a homeomorphism
   $$h_U \ : \ \pi^{-1}(U) \ \longrightarrow \ U \times \mathbb{C}^n$$
   satisfying $\pi = p \circ h_U$, where $p : U \times \mathbb{C}^n \to U$ is the projection.

   (b) The restriction of $h_U$ to every fiber is an isomorphism of complex vector spaces.

2. A complex vector bundle $\pi : E \to X$, over a Riemann surface $X$, is endowed with a holomorphic structure by specifying an open covering $\mathcal{U} = (U_i)_{i \in I}$ of $X$, and trivialization maps $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$, satisfying the following condition. For every $i$ and $j$, the natural homeomorphism (see diagram (1))
   $$h_i \circ h_j^{-1} \ : \ (U_i \cap U_j) \times \mathbb{C}^n \ \longrightarrow \ (U_i \cap U_j) \times \mathbb{C}^n$$
   is given by $(p, v) \mapsto (p, g_{ij}(p) \cdot v)$, where
   $$g_{ij} \ : \ U_i \cap U_j \ \longrightarrow \ GL(n, \mathbb{C})$$
   is a holomorphic map. (The map $g_{ij}$ is a matrix of complex valued functions, and it is holomorphic, if the entries of the matrix are holomorphic functions).

$$
\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "h_i"] & U_i \times \mathbb{C}^n \\
 & (U_i \cap U_j) \times \mathbb{C}^n \arrow[u, phantom, "\cup"] \\
\pi^{-1}(U_i \cap U_j) \arrow[uu, phantom, "\cup"] \arrow[ur, "h_{i|}"] \arrow[dr, "h_{j|}"'] & \\
 & (U_i \cap U_j) \times \mathbb{C}^n \arrow[uu, "h_i \circ h_j^{-1}"'] \\
\pi^{-1}(U_j) \arrow[uu, phantom, "\cap"] \arrow[r, "h_j"] & U_j \times \mathbb{C}^n \arrow[u, phantom, "\cap"]
\end{tikzcd} \tag{1}
$$

Two such data $\{(U_i, h_i) : i \in I\}$ and $\{(U_j, h_j) : j \in J\}$ are said to be *compatible* or *equivalent*, if their common refinement $\{(U_k, h_k) : k \in I \cup J\}$ satisfies the above condition as well. A *holomorphic structure* on a complex vector bundle is an equivalence class of such data.

1. (a) Let $X$ be a compact Riemann surface. Jacobi's Theorem (Section 21 in Forster) implies that the connecting homomorphism in problem 16.4

$$Div(X) \to H^1(X, \mathcal{O}^*)$$

is surjective. Use this fact, together with problem 16.4, to conclude that $H^1(X, \mathcal{O}^*)$ is isomorphic to the *divisor class group*, the quotient of $Div(X)$ by the group of principal divisors.

(b) Let $X = \mathbb{P}^1$ and the divisor $D$ be $n \cdot \infty$, where $\infty$ is the point of $\mathbb{P}^1$. Calculate the image of $D \in Div(\mathbb{P}^1)$ under the connecting homomorphism $\delta : Div(\mathbb{P}^1) \to H^1(\mathbb{P}^1, \mathcal{O}^\times)$ in problem 16.4. Represent $\delta(D)$ in terms of a cocycle in $H^1(\mathcal{U}, \mathcal{O}^\times)$, where the open covering $\mathcal{U}$ consists of the two open subsets $U_0 := \mathbb{P}^1 \setminus \{\infty\}$ and $U_\infty := \mathbb{P}^1 \setminus \{0\}$.

(c) Let $\pi : L \to X$ be a holomorphic line bundle over a compact Riemann surface $X$ and $s$ a meromorphic section of $L$. Define the order $\mathrm{Ord}_p(s)$ of $s$ at a point $p \in X$ as follows. Choose a triviallization $h_U : \pi^{-1}(U) \to U \times \mathbb{C}$, over some open neighborhhod $U$ of $p$. Then $h \circ s$ is the graph of a meromorphic function $f$ on $U$. Set $\mathrm{Ord}_p(s) := \mathrm{Ord}_p(f)$. Prove that the sheaf of holomorphic sections of $L$ is isomorphic to $\mathcal{O}_X(D)$, where $D := \sum_{p \in X} \mathrm{Ord}_p(s) \cdot p$.

(d) Show that the sheaf of holomorphic sections of the line bundle, corresponding to $\delta(n \cdot \infty)$ in part 1b, is $\mathcal{O}_{\mathbb{P}^1}(-n \cdot \infty)$. *Note: The standard map from $Div(X)$ to $H^1(X, \mathcal{O}^\times)$ is $-\delta$. More precisely, the standard map is the connecting homomorphism for the short exact sequence, as in problem 16.4, but with $\beta$ in 16.4 replaced by $-\beta$. This will ensure, that the class of $\delta(D)$ represents the line bundle, whose sheaf of sections is $\mathcal{O}_X(D)$ (see Problem 2).*

2. Let $X$ be a Riemann surface. Prove that there is a group isomorphism between i) The cohomology group $H^1(X, \mathcal{O}^*)$ and ii) The Picard group $\mathrm{Pic}(X)$ of isomorphism classes of holomorphic line bundles on $X$ (with the tensor product operation). You may want to follow the following steps.

(a) Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of $X$. Define a natural map $\mathcal{L}$, from the group of cocycles $Z^1(\mathcal{U}, \mathcal{O}^*)$, to the set $\mathrm{Pic}(X)$ of isomorphism classes of line bundles. The map $\mathcal{L}$ sends a cocycle $(g_{ij})$ to the line bundle obtained (as a manifold), from the disjoint union of the open sets $\{U_i \times \mathbb{C} \ : \ i \in I\}$, by gluing $U_i \times \mathbb{C}$ to $U_j \times \mathbb{C}$ via $g_{ij}$

$$\begin{array}{ccc} U_j \times \mathbb{C} & & U_i \times \mathbb{C} \\ \cup & & \cup \\ (U_i \cap U_j) \times \mathbb{C} & \xrightarrow{g_{ij}} & (U_i \cap U_j) \times \mathbb{C} \\ (p, \lambda) & \mapsto & (p, g_{ij}(p) \cdot \lambda). \end{array}$$

Interpret the $Z^1(\mathcal{U}, \mathcal{O}^*)$ group operation as a tensor product of line bundles (explain why it deserves that name). *Note: one can define independently a tensor product operation on the set* $\mathrm{Pic}(X)$*, but it takes some writing to prove that it is well defined on the level of isomorphism classes. You will get the operation for free once you construct below a one-to-one correspondence between* $\mathrm{Pic}(X)$ *and* $H^1(X, \mathcal{O}^*)$.

(b) Prove that the map $\mathcal{L} : Z^1(\mathcal{U}, \mathcal{O}^*) \to \mathrm{Pic}(X)$ factors through $H^1(\mathcal{U}, \mathcal{O}^*)$. In other words, given a 1-cocycle $(g_{ij})$ and a 0-cochain $(f_i)$, the line bundles $\mathcal{L}(g_{ij})$ and $\mathcal{L}(g_{ij} \cdot \frac{f_j}{f_i})$) are isomorphic. *Hint: As a worm-up, show that the holomorphic line bundle, associated to a 1-coboundary, is trivial. Show first that such a line-bundle has a global non-vanishing section. Observation: Let* $(g_{ij})$ *be a cocycle in* $Z^1(\mathcal{U}, \mathcal{O}^*)$. *A collection of local functions* $s_i : U_i \to \mathbb{C}$ *glues to a global section of the line-bundle* $\mathcal{L}(g_{ij})$, *if and only if* $g_{ij}s_j = s_i$ *on* $U_i \cap U_j$, *for all* $i, j \in I$.

(c) Conclude that there is a well defined injective map $\ell : H^1(X, \mathcal{O}^*) \hookrightarrow \mathrm{Pic}(X)$. *Hint: You can either use the definition of* $H^1(X, \mathcal{O}^*)$ *as a direct limit, or use Leray's Theorem, after you prove the vanishing of* $H^1(U, \mathcal{O}^*)$, *when* $U$ *is a disk. For the latter vanishing, use the exponential sequence*

$$0 \to \mathbb{Z} \longrightarrow \mathcal{O}_U \xrightarrow{e^{2\pi i(\bullet)}} \mathcal{O}_U^* \to 0$$

*and the vanishing of* $H^i(U, \mathbb{Z})$, *for* $i > 0$.

(d) Prove that the induced map $\ell : H^1(X, \mathcal{O}^*) \to \mathrm{Pic}(X)$ is surjective. (In other words, every holomorphic line bundle is isomorphic to one coming from a cocycle in $Z^1(\mathcal{U}, \mathcal{O}^*)$, for some open covering $\mathcal{U}$).