Riemann Surfaces Homework Assignment 7 Fall 2004

1. Forster section 15 page 126 problems: 15.2 and 15.3.

2. Forster section 15 page 126 problem: 15.4

   Note: Read Theorem 10.10 page 73, Definition 10.11 of the Periods, and Theorem 10.15. Forster works out the definition of the integral along any continuous curve in great detail. He does that in order to establish a relationship with universal covers. We need here much less, namely the existence of the integral and the results mentioned above. In particular, Theorem 10.15 can be proven by a simpler argument indicated in the remark at the top of page 76 (we presented this argument in class). The definition of the integral along a continuous curve is worked out in complex Analysis courses (See Lang Ch III Sections 4 and 5 pages 110 to 118, or Ahlfors, Problem 2 page 117). The discussion in Lang and Ahlfors is for holomorphic 1-forms and for open subsets of $\mathbb{C}$, but the same argument works word by word if one considers continuous curves in any Riemann surface and replaces a holomorphic 1-form by a closed 1-form. All one needs, for the discussion in Lang and Ahlfors, is the fact, that the integral along a piecewise smooth path in a disk depends only on the end-points. For holomorphic 1-forms on the disk, this follows from Cauchy's Theorem in the disk. For smooth 1-forms, it follows from the fact that a closed smooth one-from on a disk is exact.

3. Forster page 131 problems 16.1, 16.2, 16.3. *Hints:* 16.2 is Homework 4 Problem 4 part b. For 16.1: Let $d := \deg(D)$. Use problem 1 part a in Homework 6 to prove that $\mathcal{O}_{\mathbb{P}^1}(D)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(d\infty)$ and reduce problem 16.1 to problem 3 in homework 6.

4. (a) Forster page 131 problem 16.4.

   (b) Let X be a compact Riemann surface. Jacobi's Theorem (Section 21 in Forster) implies that the connecting homomorphism in problem 16.4

   $$Div(X) \to H^1(X, \mathcal{O}^*)$$

   is surjective. Use this fact, together with problem 16.4, to conclude that $H^1(X, \mathcal{O}^*)$ is isomorphic to the *divisor class group*, the quotient of $Div(X)$ by the group of principal divisors.

   Note: $H^1(X, \mathcal{O}^*)$ is isomorphic also to the group of isomorphism classes of holomorphic line-bundles. This will be indicated in a future optional problem (which will be needed for those planning to take a continuation of this course next semester with Pedit, who will probably use the language of line-bundles, rather than sheaves, being a differential geometer).

5. Forster problem 17.5 page 146. *Hint:* Use Serre's Duality Theorem to prove first that p is a base point of the sheaf Ω_X if and only if $\dim H^0(X, \mathcal{O}_X(p)) = 2$.
   Definition: The natural map $X \to \mathbb{P}H^0(X, \Omega)^*$ is called the *canonical morphism.*

   The rest of this problem set is related to the following Theorem (do not use this Theorem; it is stated in order for you to see the "big picture"). The genus 2 case of the Theorem was worked out in problem 4 of Homework 6 (all curves of genus 2 are hyperelliptic).

   Theorem: *Let X be a Riemann surface of genus $g \geq 2$. Then precisely one of the following two statements holds:*

   (a) The canonical morphism is a holomorphic embedding as a smooth algebraic curve of degree $2g-2$ in $\mathbb{P}^{g-1}$.

   (b) X is hyperelliptic and the canonical morphism factors through the branched 2-sheeted covering $X \to \mathbb{P}^1$ and a holomorphic embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}H^0(X, \Omega)^*$.

6. (a) (warm-up) Let (s,t) be the homogeneous coordinates on $\mathbb{P}^1$, and $(x_0, x_1, \ldots, x_d)$ those on $\mathbb{P}^d$. Set $\infty := [0,1] \in \mathbb{P}^1$. Let $\psi : \mathbb{P}^1 \to \mathbb{P}^d$ be the *d-uple embedding* given by

$$[s,t] \quad \mapsto \quad [s^d, s^{d-1}t^1, \ldots, s^{d-k}t^k, \ldots, t^d].$$

(see problem 5 part d in Homework 2 for the case $d = 3$). Check that for a suitable choice of coordinates on $|d\infty|^* := \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d\infty))^*$, the map ψ is the map $\mathbb{P}^1 \to |d\infty|^*$ of problem 4 part d in Homework 4 (or equivalently in Forster's Theorem 17.22).

(b) **Holomorphic embeddings of a Riemann surface C in projective spaces all come from pairs (D, V), of a divisor D and a subspace $V \subset H^0(C, \mathcal{O}_C(D))$.** Let C be a compact Riemann surface and $\psi : C \hookrightarrow \mathbb{P}^n$ a holomorphic embedding (Forster 17.20). Let $Y \subset \mathbb{P}^n$ be a hypersurface, whose homogeneous ideal is generated by a homogeneous polynomial $F(x_0, x_1, \ldots, x_n)$ of degree d. Define the *intersection multiplicity* $I_{\psi(p)}(\psi(C), Y)$ of $\psi(C)$ and Y at the point $\psi(p)$ as follows. Choose an affine chart $U_i := (x_i \neq 0) \subset \mathbb{P}^n$ containing $\psi(p)$ and set

$$I_{\psi(p)}(\psi(C), Y) \ := \ \begin{cases} \infty & \text{if} \quad \psi(C) \subset Y, \\ ord_p\left(\psi^*\left[F(x_0, \ldots, x_n)/(x_i)^d\right]\right) & \text{if} \quad \psi(C) \not\subset Y \end{cases}$$

The definition is clearly independent of the choice of the affine chart U_i containing p. (Compare with problem 7d in homework 3 when n=2).

Assume now that $\psi(C)$ is not contained in any hyperplane and $Y \subset \mathbb{P}^n$ is a linear hyperplane ($\deg(F) = 1$). Set $D := \sum_{p \in C} I_{\psi(p)}(\psi(C), Y) \cdot p \in Div(C)$.

i. Prove that the pullback homomorphism

$$\psi^* \ : \ \mathrm{span}_{\mathbb{C}}\left\{\frac{x_0}{F}, \ldots, \frac{x_n}{F}\right\} \quad \longrightarrow \mathcal{M}(C)$$

is injective and its image V is contained in $H^0(C, \mathcal{O}_C(D))$. *Note:* The standard notation for the domain of the above homomorphism is $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(Y))$.

ii. Prove that $\mathcal{O}_C(D)$ is generated by its global sections in V, i.e.,

$$\dim[V \cap H^0(C, \mathcal{O}_C(D-p))] = \dim(V) - 1, \quad \text{for all} \ \ p \in C.$$

iii. Prove that the morphism

$$\begin{aligned} \varphi \ : \ C \ &\to \ \mathbb{P}V^*, \\ p \ &\mapsto \ V \cap H^0(C, \mathcal{O}_C(D-p)) \end{aligned}$$

is precisely the given embedding ψ, when the homogeneous coordinates on $\mathbb{P}V^*$ correspond to the basis $\{\psi^*(x_0/F), \ldots, \psi^*(x_n/F)\}$ of V.

7. Let X and Y be compact Riemann surfaces, $\pi : X \to Y$ a d-sheeted branched cover, f a non-zero meromorphic function on Y with divisor $(f) := \sum_{q \in Y} ord_q(f)$, and ω a meromorphic 1-form on Y with divisor $(\omega) := \sum_{q \in Y} ord_q(w)$. Given any divisor $\Delta := \sum_{i=1}^k n_i q_i$ in $Div(Y)$, set

$$\pi^*(\Delta) := \sum_{i=1}^k n_i \left(\sum_{p \in \pi^{-1}(q_i)} m_p \cdot p \right),$$

where m_p is the multiplicity of p in the fiber of π.

(a) Prove the equalities

$$(\pi^* f) = \pi^*(f) \quad \text{and} \quad (\pi^*\omega) = \pi^*(\omega) + \sum_{p \in X}(m_p - 1)p$$

of divisors on X. (The second equality was proven for exact 1-forms when part a of Problem 5 in Homework 5 was proven).

(b) Assume now that $Y = \mathbb{P}^1$ and $\deg(\pi) = 2$. Prove that the ramification divisor $\sum_{p \in X}(m_p - 1)p \in Div(X)$ is linearly equivalent to $\pi^*((g+1)\infty)$. Conclude, that Ω_X is isomorphic to $\mathcal{O}_X(\pi^*((g-1)\infty))$. *Hint:* X is uniquely determined by its branch points (by problem 3 in homework 5), so you may use your calculations in problem 6 of homework 5.

(c) Prove, under the assumptions of part 7b, that the canonical morphism $\varphi : X \to \mathbb{P}H^0(X, \Omega_X)^*$ factors via π through the $(g-1)$-uple embedding of $\mathbb{P}^1$, i.e., $\varphi = \psi \circ \pi$ for a suitable choice of coordinates on $\mathbb{P}H^0(X, \Omega_X)^*$, where ψ is given in Problem 6a. Conclude, when $g = 3$, that the image of φ is a smooth conic (degree 2) curve in $\mathbb{P}^2$.

8. (The adjunction formula for plane curves) Let C be a smooth algebraic curve of degree d in $\mathbb{P}^2$ given by $F(x, y, z) = 0$, with F irreducible, and let H be a line (curve of degree 1) in $\mathbb{P}^2$. (If $d = 1$, assume that H and C are different lines). Set $D := C \cap H := \sum_{p \in C} I_p(C, H) \cdot p \in Div(C)$. Prove the *Adjunction Formula* which states that the sheaf Ω_C is isomorphic to $\mathcal{O}_C((d-3)D)$. Use the following steps:

   (a) Let $G(x, y, z)$ be a homogeneous polynomial of degree n, such that the curve $G = 0$ does not contain C. Prove that the divisor $\sum_{p \in C} I_p(F, G) \cdot p$ in $Div(C)$ is linearly equivalent to nD. *Hint:* Let $L(x, y, z)$ be the degree 1 polynomial corresponding to H, consider the rational function $(G/L^n)_{|C}$, and use part (d) of problem 7 in Homework 3.

   (b) Let $\pi : C \to \mathbb{P}^1$ be the projection of the curve C from a point, given in problem 4 of homework 3. Prove that $\sum_{p \in C}(m_p - 1)p$ is linearly equivalent to $(d-1)D$. *Hint:* Use part b of problem 4 in homework 3).

   (c) Prove the adjunction formula using problem 7a.

9. (A Riemann surface of genus 3 is either hyperelliptic or a plane quartic)

   (a) Let C be a smooth algebraic curve of degree 4 in $\mathbb{P}^2$. Use Forster Remark 17.10 to conclude that $H^0(C, \Omega_C)$ is 3-dimensional. Prove that there exists a unique projective linear isomorphism $f : \mathbb{P}^2 \to \mathbb{P}H^0(C, \Omega_C)^*$, such that $f \circ \iota = \varphi$, where $\iota : C \hookrightarrow \mathbb{P}^2$ is the embedding given and $\varphi : C \to \mathbb{P}H^0(C, \Omega_C)^*$ is the canonical morphism. In other words, φ is equal to ι, for a suitable choice of coordinates on $\mathbb{P}H^0(C, \Omega_C)^*$. *Hint:* Use the adjunction formula (problem 8) and problems 5 and 6b.

   (b) Use problem 7c to conclude that C is **not** hyperelliptic (there does not exist a proper holomorphic map $\pi : C \to \mathbb{P}^1$ of degree 2).

Note: More generally, "most" Riemann surfaces of genus $g \geq 3$ are non-hyperelliptic. For example, an argument similar to the one you used above shows, that if $Q, F \in \mathbb{C}[x_0, x_1, x_2, x_3]$ are homogeneous polynomials of degrees 2 and 3 respectively, and $X := V(Q, F)$ is a smooth algebraic curve, then X has genus 4 and the embedding $X \subset \mathbb{P}^3$ is the canonical embedding, so X is not hyperelliptic by problem 7c.