Riemann Surfaces Homework Assignment 6 Fall 2004

1. (a) Forster section 10 problem: 10.4. *Hint*: See Corollary 10.22.

   (b) Let X be a compact Riemann surface and D_1, D_2 two divisors, which are linearly equivalent (i.e., $D_1 - D_2$ is the divisor of a meromorphic function on X). Prove the equality $\dim H^0(X, \mathcal{O}_X(D_1)) = \dim H^0(X, \mathcal{O}_X(D_2))$. (In fact, the two sheaves are isomorphic in the sense of problem 4a below).

   (c) Let X be a compact Riemann surface of genus $g \geq 1$ and D a divisor of degree 1 on X. Prove that $\dim H^0(X, \mathcal{O}_X(D))$ is either 0 or 1. *Hint:* In case $\dim H^0(X, \mathcal{O}_X(D)) > 0$, show that D is linearly equivalent to p, for some point $p \in X$.

   (d) Let X be a compact Riemann surface, D a divisor on X, and p a point of X. Prove that $H^0(X, \mathcal{O}_X(D-p))$ is a subspace of $H^0(X, \mathcal{O}_X(D))$ and the difference $\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \mathcal{O}_X(D-p))$ is either 0 or 1. Conclude, that if the genus of X is ≥ 1, then $\dim H^0(X, \mathcal{O}_X(D)) \leq \deg(D)$.

2. Forster section 12 problems: 12.1, 12.2, and 12.3. Problem 12.2 proves the finite dimensionality of $H^1(X, \mathbb{C})$ for a compact Riemann surface X using a technical trick, which avoids a choice of a nice covering. The notation $V \Subset U$ denotes that the closure of V is a compact subset of U. Note that a simpler proof would follow if we knew that we can choose a covering of X by finitely many simply connected open subsets $\{U_i\}_{i \in I}$, such that $U_i \cap U_j$ has finitely many connected components, for all i and j. (The existence of such a covering follows easily from standard facts about the topology of compact oriented two dimensional manifolds).

3. Set $U_0 := \mathbb{P}^1 \setminus \{\infty\}$, $U_1 := \mathbb{P}^1 \setminus \{0\}$, and let $\mathcal{U} := (U_0, U_1)$ be the corresponding open covering of $\mathbb{P}^1$. Calculate $\dim H^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(n\infty))$ and $\dim H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(n\infty))$, for all integers n. Above $n\infty$ is the divisor on $\mathbb{P}^1$. Conclude the equality: (A special case of Serre's Duality Theorem)

$$\dim H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}(n\infty)) \quad = \quad \dim H^0(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1}((-2-n)\infty))$$

   Hint: See the proof of Forster's Theorem 13.5. We have used the same argument in class to calculate $H^1(\mathcal{U}, \Omega)$.

4. (a) Let X be a compact Riemann surface and w a non-zero meromorphic 1-form on X with divisor $K := (\omega)$ (see Problem 4 in homework 5 for the definition of the divisor of a 1-form). Show that the sheaf $\mathcal{O}_X(K)$ is isomorphic to the sheaf Ω of holomorphic 1-forms on X. In other words, w induces isomorphisms $f_U : \mathcal{O}_X(K)(U) \to \Omega(U)$, of vector spaces, for any open subset U of X, and these commute with the restriction homomorphisms of the two sheaves, $\rho_V^U \circ f_U = f_V \circ \rho_V^U$, for every two open subsets, such that $V \subset U$.

   (b) (Every Riemann surface of genus 2 is hyperelliptic) Let X be a compact Riemann surface. Remark 17.10 in Forster states, that $\dim H^0(X, \Omega)$ is always equal to the genus of X. You have seen an explicit proof of this fact in Homework 5 problem 6 part (d), when X is a hyperelliptic Riemann surface. Assume now that X is a compact Riemann surface of genus 2 and $\dim H^0(X, \Omega) = 2$. Prove that there is a two-sheeted holomorphic covering map $\pi : X \to \mathbb{P}^1$ branched along 6 points. *Hint:* Let w be a non-zero holomorphic 1-form with divisor $K := (w)$. Show first, that $\dim H^0(X, \mathcal{O}_X(K-p)) = 1$, for every point $p \in X$. Then let π be the map $X \to |K|^* := \mathbb{P}H^0(X, \mathcal{O}_X(K))^*$ defined in Homework 4 problem 4d.