1. Forster section 9 page 69 problems: 9.1, 9.3.
2. Let $\tau$ be a complex number with $\operatorname{Re}(\tau)>0, \Gamma:=\operatorname{span}_{\mathbb{Z}}\{1, \tau\}$ the lattice in $\mathbb{C}$, and $Y:=$ $\mathbb{C} / \Gamma$ the corresponding torus. Show that there are precisely three pairwise-non-equivalent connected unramified double covers $p_{i}: X_{i} \rightarrow Y, i=1,2,3$. Show furtheremore that $X_{i}$ is a compact torus and find a basis for its lattice. Note: Two covering maps $p_{i}$ and $p_{j}$ are equivalent, if there exists a biholomorphic map $f: X_{i} \rightarrow X_{j}$ satisfying $p_{j} \circ f=p_{i}$. Hint: Use Theorem 4.6 in Forster to reduce the discussion to a topological one, forgetting about the complex structure. Prove that if $p_{i}: X_{i} \rightarrow Y, i=1,2$, are equivalent coverings, $q_{i} \in X_{i}$, and $p_{1}\left(q_{1}\right)=p_{2}\left(q_{2}\right)$, then $p_{1_{*}}\left(\pi_{1}\left(X_{1}, q_{1}\right)\right)=p_{2_{*}}\left(\pi_{1}\left(X_{2}, q_{2}\right)\right)$. Then consider index two subgroups of the fundamental group of $Y$, the identification of $\Gamma$ with $\pi_{1}(Y, 0)$, and Theorem 5.9 and exercise 5.2 in Forster.
3. (Double covers of $\mathbb{P}^{1}$ are determined uniquely by their branch points). Let $X$ and $Y$ be two compact Riemann surfaces and $\pi: X \rightarrow Y$ a proper holomorphic map of degree 2. Let $A \subset X$ be the set of ramification points and $B:=\pi(A) \subset Y$ the set of branch points of $\pi$. Recall that $\pi$ restricts to $X \backslash A$ as a covering of $Y \backslash B$, which is Galois by Exercise 5.7 page 39 in Forster. Let $\sigma$ be the Deck transformation interchanging the points in the unramified fibers of $\pi$. Then $\sigma$ extends to a holomorphic map $\sigma: X \rightarrow X$, fixing all the ramification points, by Forster Theorem 8.5. Denote by $\sigma^{*}: \mathcal{M}(X) \rightarrow \mathcal{M}(X)$ the pullback homomorphism sending a meromorphic function $f$ on $X$ to $f \circ \sigma$. Define $\pi^{*}: \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$ similarly.
(a) Show that if a meromorphic function $f \in \mathcal{M}(X)$ is sent to itself by the Galois involution, i.e., $\sigma^{*}(f)=f$, then $\operatorname{ord}_{p}(f)$ is even for all $p \in A$. Similarly, if $\sigma^{*}(f)=-f$, then $\operatorname{ord}_{p}(f)$ is odd at all the ramification points.
(b) Show that $\sigma^{*}(f)=f$, if and only if $f=\pi^{*}(g)$ for some $g \in \mathcal{M}(Y)$.
(c) Choose a point $p \in X$ and let $D$ be the divisor $p+\sigma(p) \in \operatorname{Div}(X)$. Recall that $H^{0}\left(X, \mathcal{O}_{X}(n D)\right)$ is the subspace of $\mathcal{M}(X)$ consisting of functions $f$, which are holomorphic on $X \backslash\{p, \sigma(p)\}$ and satisfying $\operatorname{ord}_{p}(f)+n \geq 0$ and $\operatorname{ord}_{\sigma(p)}(f)+n \geq 0$, if $p \neq \sigma(p)$, and $\operatorname{ord}_{p}(f)+2 n \geq 0$, if $p=\sigma(p)$. Show that $H^{0}\left(X, \mathcal{O}_{X}(n D)\right)$ is $\sigma^{*}$ invariant and denote the +1 and -1 eigenspaces by $H^{0}\left(X, \mathcal{O}_{X}(n D)\right)^{+}$and $H^{0}\left(X, \mathcal{O}_{X}(n D)\right)^{-}$. Prove the equality: $\quad H^{0}\left(X, \mathcal{O}_{X}(n D)\right)^{+}=\pi^{*} H^{0}\left(X, \mathcal{O}_{Y}(n \pi(p))\right.$.
Use the theorem in Homework 4 Problem 4, about the dimension of $H^{0}\left(X, \mathcal{O}_{X}(n D)\right)$, to show that $H^{0}\left(X, \mathcal{O}_{X}(n D)\right)^{-}$is non-trivial, for all $n>\max \left\{0,2 g_{Y}-2+b\right\}$, where $g_{Y}$ is the genus of $Y$ and $b$ is the number of branch points. Hint: Use also the Riemann-Hurwitz formula (Homework 3 Problem 4).
(d) Prove that $\mathcal{M}(X)$ is equal to $\left(\pi^{*} \mathcal{M}(Y)\right)[f]$ for some $f \in \mathcal{M}(X)$ satisfying $\sigma^{*}(f)=-f$. Conclude that $\mathcal{M}(X)$ is the field extension obtained from $\mathcal{M}(Y)$ by adjoining a root $f$ of the polynomial $T^{2}-h \in \mathcal{M}(Y)[T]$, where $T$ is a transcendental variable. Furthermore, $\operatorname{ord}_{y}(h)$ is even for $y \in Y \backslash B$ and odd for $y \in B$. Observe that the map $(\pi, f): X \longrightarrow$ $Y \times \mathbb{P}^{1}$ is injective away from the zeroes and poles of $f$.
(e) Assume now that $Y=\mathbb{P}^{1}$. Show that $h$ in part 3d can be chosen to have zeroes and poles of order 1. Conclude that given two holomorphic branched double covers $\pi_{i}: X_{i} \rightarrow \mathbb{P}^{1}$, having the same branch locus $B$, there exists a biholomorphic map $\varphi: X_{1} \rightarrow X_{2}$ satisfying $\pi_{2} \circ \varphi=\pi_{1}$. (You may use Forster Exercise 8.2 to shorten your argument).
4. Let $X$ be a compact Riemann surface and $\omega$ a non-zero meromorphic 1-form on $X$. Define $\operatorname{ord}_{p}(\omega)$, for $p \in X$, as follows. If $z$ is a local coordinate on an open subset $U$ of $X$ containing
$p$, and $f$ is the meromorphic function on $U$ satisfying $\omega=f d z$, then set $\operatorname{ord}_{p}(\omega):=\operatorname{ord}_{p}(f)$. Clearly, $\operatorname{ord}_{p}(\omega)$ is independent of the choice of $z$. Denote by $(\omega)$ the divisor

$$
(\omega):=\sum_{p \in X} \operatorname{ord}_{p}(\omega) \cdot p .
$$

(a) Show that if $\omega_{1}$ and $\omega_{2}$ are two non-zero meromorphic 1-forms, then the divisors ( $\omega_{1}$ ) and $\left(\omega_{2}\right)$ are linearly equivalent. Conclude, that $\operatorname{deg}\left(\omega_{1}\right)=\operatorname{deg}\left(\omega_{2}\right)$. Recall:
Definition: Two divisors $D_{1}$ and $D_{2}$ are linearly equivalent, if there exists a meromorphic function $f$, such that $D_{2}-D_{1}=(f)$, where $(f)$ is the divisor of $f$.
(b) Definition: Let $\Omega_{X}$ be the sheaf of holomorphic 1-froms on $X$. The common degree, of all the divisors of meromorphic 1 -forms on $X$, is denoted by $\operatorname{deg}\left(\Omega_{X}\right)$.
Prove that $\operatorname{deg}\left(\Omega_{\mathbb{P}^{1}}\right)=-2$ and for a compact torus $X$ show that $\operatorname{deg}\left(\Omega_{X}\right)=0$. (Do it here directly, but see also problem 5).
5. The degree of the sheaf of holomorphic 1-forms in terms of the genus:

Let $\pi: X \rightarrow Y$ be an $n$-sheeted branched covering of a compact Riemann surface $Y$, $x_{1}, x_{2}, \ldots, x_{k}$ the ramification points of $\pi$, and $m_{i}$ the multiplicity of $x_{i}$ in the fiber of $\pi$. Assume, that there exists a non-constant meromorphic function $f$ on $Y$ (this is always true and follows from the Theorem in Problem 4 of Homework 4, which will be proven later).
(a) Use the meromorphic 1-form $d f$ and its pullback $\pi^{*} d f$ to prove the equality:

$$
\begin{equation*}
\operatorname{deg}\left(\Omega_{X}\right)=n \cdot \operatorname{deg}\left(\Omega_{Y}\right)+\sum_{i=1}^{k}\left(m_{i}-1\right) \tag{1}
\end{equation*}
$$

where $\operatorname{deg}\left(\Omega_{Y}\right)$ is defined in Problem 4.
(b) Let $C \subset \mathbb{P}^{2}$ be a smooth projective plane curve of degree $d$. Prove that $\operatorname{deg}\left(\Omega_{C}\right)=$ $d(d-3)$. (Do not use the Riemann-Hurwitz formula). Hint: Use parts b and c of problem 4 in Homework 3 and formula (1) above.
(c) Let $\pi$ be a non-constant meromorphic function on a compact Riemann surface $X$. Use the Riemann-Hurwitz formula (problem 4 in Homework 3) to prove the equality $\operatorname{deg}\left(\Omega_{X}\right)=$ $2 g_{X}-2$, where $g_{X}$ is the genus of $X$.
6. (a) Let $g$ be a non-negative integer, $\left\{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{2 g+1}\right\}$ distinct complex numbers, and $X_{0}$ the affine algebraic curve in $\mathbb{C}^{2}$ given by $y^{2}-\prod_{i=1}^{2 g+1}\left(x-\lambda_{i}\right)=0$.
Modify the construction in class to obtain a compact Riemann surface $X$, containing $X_{0}$, and a branched double cover $\pi: X \rightarrow \mathbb{P}^{1}$, whose restriction to $X_{0}$ is equal to the restriction of the function $x$ from $\mathbb{C}^{2}$ to $X_{0}$. Hint: You will be forced to let $\infty \in \mathbb{P}^{1}$ be a branch point of $\pi$.
(b) Denote the restriction of $x$ and $y$ to $X$ by $x$ and $y$ as well. Determine the zeroes and poles, and their multiplicities, for the following meromorphic one forms: i) $\frac{d y}{x}$, ii) $\frac{d x}{y}$ (you should get that the last one is holomorphic). For each of the above forms show that the difference, between the number of zeroes and the number of poles, counted with multiplicities, is equal to $2 g-2$.
(c) Let $\omega$ be a holomorphic 1-form on $X$ satisfying $\sigma^{*}(w)=\omega$, where $\sigma \in \operatorname{Deck}\left(X / \mathbb{P}^{1}\right)$ is the Galois involution. Prove that $\omega=0$. Hint: See problem 3b.
(d) Prove that every holomorphic 1-form on $X$ is of the form $f(x) d x / y$, where $f$ is a polynomial of degree $\leq g-1$. Conclude that the vector space $H^{0}\left(X, \Omega_{X}\right)$ of global holomorphic 1 -forms on $X$ is $g$-dimensional.
(e) Calculate the residues for each of the 1-forms in part 6b.

