Riemann Surfaces Homework Assignment 4 Fall 2004

1. Kirwan Problems 4.3 and 4.5 page 110 (From HW3, nobody signed-up last time).

2. **Definitions 5.4 and 5.5 in Forster:** Let $p : Y \to X$ be a covering map. A *covering* (or *deck* or *Galois*) *transformation* of p is a fiber preserving homeomorphism $f : Y \to Y$ (i.e., such that $p \circ f = p$). The set $Deck(Y/X)$ (or $Gal(Y/X)$) of all deck transformations is a subgroup of the group of homeomorphisms from Y to itself. We say that the covering p is *Galois*, if $Deck(Y/X)$ acts transitively on each fiber of p.

   Fix a point $y_0 \in Y$. Observe, that every covering transformation $f : Y \to Y$ is a lift of p in the sense of Theorem 4.17 and is hence determined uniquely by the image $f(y_0)$. In other words, the map $Deck(Y/X) \longrightarrow p^{-1}(p(y_0)) \subset Y$, sending f to $f(y_0)$, is always injective and it is surjective if and only if the covering is Galois.

   Forster section 5 page 38 problems:

   (a) 5.4 (a continuation of Problem 1.5 (a) page 9).
   Hint: Prove first the existence of a function F making the diagram commutative. Then show that the derivative F' is doubly periodic. This exercise depends on the material from section 4. The only exception is the isomorphism $Deck(\mathbb{C}/\Gamma \stackrel{f}{\to} \mathbb{C}/\Gamma') \cong \Gamma'/\alpha\Gamma$. The latter isomorphism follows from problem 5.2 (which is a standard theorem when one studies covering spaces).

   (b) 5.6 *Hint:* All you need from section 5 is the Definition above. Extend p as a branched cover $\bar{p} : \mathbb{C} \to \mathbb{C}$ and find all its ramification points and its branch points. Show that if p is Galois then all the points in any given fiber of $\bar{p}$ must have the same ramification index (consider the connected components of the inverse image under p of a punctured disk centered at one of the branch points).

   (c) 5.7 (All you need from section 5 is the above Definition.)

3. Forster section 6 page 44 problems: 6.1, 6.2

4. Let Σ be a compact Riemann surface of genus g. (We will define the genus later in class, we are relying now on your topology background). Recall the following
   Definitions: The group of divisors $Div(\Sigma)$ is the free abelian group generated by points of Σ. Specifying a divisor $D \in Div(\Sigma)$ is equivalent to specifying a function $\tilde{D} : \Sigma \to \mathbb{Z}$, which vanishes at all but finitely many points (we think of $\tilde{D}(p)$ as the integer coefficient of the point p, so that $D = \sum_{p\in\Sigma} \tilde{D}(p) \cdot p$). The latter description introduces the partial ordering on $Div(\Sigma)$, where $D_1 \geq D_2$ if $\tilde{D}_1(p) \geq \tilde{D}_2(p)$, for all $p \in \Sigma$. The divisor $(f) \in Div(\Sigma)$ of a meromorphic function f on Σ, which is not identically zero, is

$$(f) \;:=\; \sum_{p\in\Sigma} ord_p(f) \cdot p \;=\; \sum_{p\in f^{-1}(0)} ord_p(f) \cdot p \;-\; \sum_{p\in f^{-1}(\infty)} ord_p\left(\frac{1}{f}\right) \cdot p.$$

   Fix a divisor $D \in Div(\Sigma)$. The vector space $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ is the subspace of the field $\mathcal{M}(\Sigma)$ of meromorphic functions on Σ given by

$$H^0(\Sigma, \mathcal{O}_\Sigma(D)) \;:=\; \{f \in \mathcal{M}(\Sigma) \;:\; f = 0 \quad \text{or} \quad (f) + D \geq 0\},$$

   where the latter 0 is the zero divisor. If, for example, $f \in H^0(\Sigma, \mathcal{O}_\Sigma(3p - 2q))$, $p, q \in \Sigma$, $p \neq q$, then f vanishes to order at least 2 at q and may have a pole at p but the (absolute value of) its order is at most 3. The *degree* $\deg(D)$ of a divisor is the sum $\sum_{p\in\Sigma} \tilde{D}(p)$.

   The following is a corollary of two of the main Theorems in this course, the Riemann-Roch and Serre's Duality Theorems:

Theorem: Let Σ be a compact Riemann-surface of genus g and $D \in Div(\Sigma)$ a divisor of degree d. Then $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ is *finite dimensional* and

$$\dim H^0(\Sigma, \mathcal{O}_\Sigma(D)) \quad \geq \quad d+1-g.$$

Furthermore, equality holds above if $d \geq 2g-1$.

(a) Let $X := \mathbb{C}/\Gamma$ be a compact complex torus, where $\Gamma := \text{span}_{\mathbb{Z}}\{w_1, w_2\}$ is a lattice spanned by two complex numbers linearly independent over $\mathbb{R}$. Let $p_0 \in X$ be its identity point. Calculate $\dim H^0(X, \mathcal{O}_X(dp_0))$, for all $d \in \mathbb{Z}$. Prove it without using the above Theorem. *Hint:* 1) For $d < 0$: Recall, that the degree of the divisor (f) of a meromorphic function is always 0. 2) For $d \geq 0$: Use induction and Corollary 2.8 in Forster to prove that $H^0(X, \mathcal{O}_X(dp_0))$ is spanned by monomials $\mathcal{P}^i(\mathcal{P}')^j$, where i, j are non-negative integers satisfying $2i+3j \leq d$, and $\mathcal{P} \in \mathcal{M}(X)$ is the meromorphic function on X corresponding to the Weierstrass $\mathcal{P}$-function on $\mathbb{C}$. Furthermore, we may restrict j to be 0 or 1.

(b) Conclude that sending x to $\mathcal{P}$ and y to $\mathcal{P}'$ we get a *surjective* homomorphism

$$\mathbb{C}[x,y] \quad \rightarrow \quad \mathcal{O}(X \setminus \{p_0\}) \cap \mathcal{M}(X) \tag{1}$$

onto the space of meromorphic functions, which are holomorphic on $X \setminus \{p_0\}$. Find a basis for $H^0(X, \mathcal{O}_X(dp_0))$, for $1 \leq d \leq 6$.

(c) The kernel of (1) is a prime ideal generated by a cubic polynomial of the form $f(x,y) = y^2 - [a_3x^3 + a_1x + a_0]$. Furthermore, $a_3 = 4$. *Hint:* Recall that $\mathcal{P}$ is even. *Note:* For an elementary proof, not relying on the above Theorem, see Lemma 5.17 in Kirwan.

(d) Let Σ be a compact Riemann surface of genus g and D a divisor of degree $d \geq 2g$. For every point $p \in \Sigma$, $H^0(\Sigma, \mathcal{O}_\Sigma(D-p))$ is a subspace of $H^0(\Sigma, \mathcal{O}_\Sigma(D))$ of codimension 1, hence the kernel of a linear functional, unique up to a constant factor, hence a point in the projective space $|D|^* := \mathbb{P}[H^0(\Sigma, \mathcal{O}_\Sigma(D))^*]$ associated to the dual vector space. We get a map

$$\varphi_D \; : \; \Sigma \quad \longrightarrow \quad |D|^*. \tag{2}$$

Show, that φ_D is injective, for $d \geq 2g+1$. *Hint:* Use the above Theorem. Observe that for $p \neq q$, $H^0(\Sigma, \mathcal{O}_\Sigma(D-p)) \cap H^0(\Sigma, \mathcal{O}_\Sigma(D-q)) = H^0(\Sigma, \mathcal{O}_\Sigma(D-p-q))$.

(e) Let D be the divisor $2p_0$ on the torus X in part 4a. Show that φ_D is a degree 2 holomorphic map onto $\mathbb{P}^1$. *Hint:* Let $\{e_0, e_1\}$ be the basis of $H^0(X, \mathcal{O}_X(2p_0))^*$ dual to the basis $\{1, \mathcal{P}\}$. Use the basis $\{e_0, e_1\}$ to identify $H^0(X, \mathcal{O}_X(2p_0))^*$ with $\mathbb{C}^2$ and $|D|^*$ with $\mathbb{P}^1$. What is the composition of φ_D with the meromorphic function x_1/x_0 on $\mathbb{P}^1$ (where (x_0, x_1) are the homogeneous coordinates on $\mathbb{P}^1$)?

(f) Let D be the divisor $3p_0$ on the torus X. Show that φ_D maps the torus X injectively into $\mathbb{P}^2$ and the image $\varphi_D(X)$ is contained in a cubic curve C.

(g) Prove that the cubic curve C in part 4f is smooth and $\varphi_D : X \to C$ is surjective. *Hint:* Use Kirwan problem 4.5 (see problem 1 above) and Forster Theorem 2.7. A more explicit proof of smoothness is in Kirwan Lemma 5.20. We will later prove more generally, that the image of the map φ_D in (2) is a smooth projective curve, when $d \geq 2g+1$ (Forster Theorem 17.22).