Riemann Surfaces Homework Assignment 3 Fall 2004

1. Forster section 1 page 8 problems: 1.4, 1.5 (also make sure you know 1.2 from your complex analysis class; otherwise, do it). (Problem 1.5 will be continued in the future problem 5.4)
   Correction: In problem 1.4 the matrix A should be in $SL(2,\mathbb{Z})$ and not in $SL(2,\mathbb{C})$ as indicated in old printings of the book.

2. Forster section 2 page 13 problems: 2.1, 2.4 (For 2.1 see Proposition 5.10 and Lemma 5.17 in Kirwan or your favorite complex analysis book)

3. Forster section 4 page 30 problems: 4.5

4. Kirwan's Problem 4.2 page 110 and a sharpening of Lemma 4.7 in Kirwan. Let $C = V(F(x,y,z))$ be a smooth curve of degree d in $\mathbb{P}^2$. Assume that $p_0 := [0,1,0]$ does not belong to C. Let $\pi : C \to \mathbb{P}^1$ be the projection from p_0. In other words, π sends $p = [a,b,c] \in C$ to the point $[a,0,c]$ of intersection of the line $y = 0$ with the line ℓ_{pp_0} through p and p_0. We identify $\mathbb{P}^1$ with the line $(y = 0)$.

   (a) Let $p = [a,b,c] \in C$ be a point, such that $c \neq 0$ and $\partial F/\partial x(p) = 0$. Show that π is unramified at p. *Hint:* Use first Euler's relation to conclude that $\partial F/\partial y(p) \neq 0$.

   (b) Prove that the multiplicity $m_p(\pi)$ of $p \in C$ in the fiber of π satisfies the equality

   $$m_p(\pi) \quad = \quad I_p(C, \partial F/\partial y) + 1. \tag{1}$$

   The number $m_p(\pi) - 1$ is called the *ramification index* of π at p. Prove the equality

   $$\sum_{p \in C} [m_p(\pi) - 1] \quad = \quad d(d-1). \tag{2}$$

   Hint for (1): We may assume that $p = [a,b,c]$, where $c \neq 0$, possibly after interchanging x and z. Treat separately the case of part 4a. When $\partial F/\partial x(p) \neq 0$, use $\bar{y} := y/z$ as a coordinate on C, calculate $\partial \bar{x}/\partial \bar{y}$ implicitly using $F(\bar{x}, \bar{y}, 1) = 0$, and use the property of intersection multiplicities in problem 7 equation (5).

   (c) Prove the equality $m_p(\pi) = I_p(C, \ell_{pp_0})$. Conclude the equality

   $$\sum_{p \in \pi^{-1}(q)} m_p(\pi) \quad = \quad d, \tag{3}$$

   for every point $q := [a,0,c] \in \mathbb{P}^1$. In other words, there are d points in every fiber of π, counted with multiplicities. We say that π has *degree* d.

   (d) The Degree-Genus formula states, that a smooth projective curve of degree d in $\mathbb{P}^2$ has genus $(d-1)(d-2)/2$ (Theorem 4.19 in Kirwan). Let C_1 and C_2 be two compact complex surfaces of genus g_1 and g_2 and $\pi : C_1 \to C_2$ a non-constant holomorphic map. The Riemann-Hurwitz formula (Remark 4.23 in Kirwan or the Theorem page 140 in Forster) states,

   $$(2g_1 - 2) \quad = \quad \deg(\pi)(2g_2 - 2) + \sum_{p \in C_1} [m_p(\pi) - 1].$$

   Prove the Degree-Genus formula using the Riemann-Hurwitz formula and equations (2) and (3). We will prove the Riemann-Hurwitz formula in class.

(e) Find the ramification points and their index for the map π and the curve $F(x,y,z) = x^d + y^d + z^d, \quad d \geq 1$.

5. Kirwan Problem 4.5 page 110.

6. Kirwan Problem 5.18 page 142.

7. This problem sketches the proof of the property of intersection multiplicities missing in Kirwan's book (See equation (5) below). We used this property in the proof, that a smooth projective curve of degree d in $\mathbb{P}^2$ has $3d(d-2)$ flexes, counted with multiplicities. It was an essetial part of the proof of the equality

$$I_p(C, T_pC) \quad = \quad 2 + I_p(F, \mathcal{H}_F),$$

where C is given by $F(x,y,z) = 0$ and $\mathcal{H}_F$ is the Hessian of F. (See also problem 4b.)

Motivation: Let $C = V(F(x,y))$ and $D = V(G(x,y))$ be affine curves in $\mathbb{C}^2$, where F and G are non-constant polynomials in $\mathbb{C}[x,y]$ having no irreducible component in common. The quotient ring $\mathbb{C}[x,y]/(F,G)$ is then a finite dimensional (Artinian) ring and it decomposes as a product $\prod_{p \in C \cap D} R_p$ of a finite set of rings, each associated to a point $p \in C \cap D$. We would like to prove, that the intersection multiplicity $I_p(C,D)$ is equal to the dimension $\dim(R_p)$. One can "eliminate" all factors except R_p by localizing $\mathbb{C}[x,y]/(F,G)$ at the maximal ideal associated to p. Localization commutes with taking quotients. Hence, we we will start with localization.

Localization: Given a prime ideal $\mathfrak{p}$ of a commutative ring R, one defines the *localization* $R_{(\mathfrak{p})}$ of R at $\mathfrak{p}$ by inverting elements which do not belong to $\mathfrak{p}$. If, for example, R is an integral domain, then $R_{(0)}$ is the field K of fractions of R. For any prime ideal $\mathfrak{p}$ the localization $R_{(\mathfrak{p})}$ is the subset of K consisting of elements, which can be written as a quotient f/g, where g does not belong to $\mathfrak{p}$. $R_{(\mathfrak{p})}$ is a *local ring*, i.e., it has a unique maximal ideal and any non-unit element belongs to the maximal ideal.

(a) (This part is **optional**) Let $\mathcal{O}_p(\mathbb{C}^2)$ be the localization of $\mathbb{C}[x,y]$ at the maximal ideal $(x-a, y-b)$ of the point $p = (a,b)$. Prove that the number

$$I'_p(C,D) \quad := \quad \dim_{\mathbb{C}}[\mathcal{O}_p(\mathbb{C}^2)/(F,G)] \tag{4}$$

satisfies the affine analogue of the six axioms in Kirwan for the intersection multiplicity (just drop the requirement, that the polynomials be homogeneous). The proof of uniqueness is the same. Conclude from Kirwan's Existence proof the equality $I'_p(C,D) = I_p(\overline{C}, \overline{D})$, where $\overline{C}$ and $\overline{D}$ are the closures of C and D in $\mathbb{P}^2$. *Hint:* The verification of axioms (i), (iii), (vi) and (iv) is obvious. Use Nullstellensatz and Theorem 3.9 in Kirwan's book to prove axiom (ii). Note also that if $p = (a,b) \in \mathbb{C}^2$, then the quotient $\mathbb{C}[x,y]/(x-a, y-b)^k$, by the k-th power of the maximal ideal of p, is finite dimensional. One proves axiom (v)

$$I_p(F, GH) \quad = \quad I_p(F,G) + I_p(F,H),$$

for three non-constant polynomials $F, G, H \in \mathbb{C}[x,y]$, by showing that the following sequence is exact

$$0 \to \mathcal{O}_p(\mathbb{C}^2)/(F,H) \xrightarrow{\psi} \mathcal{O}_p(\mathbb{C}^2)/(F,GH) \xrightarrow{\varphi} \mathcal{O}_p(\mathbb{C}^2)/(F,G) \to 0,$$

where φ is the quotient homomorphism and ψ is multiplication by G.

(b) Let C be an irreducible curve in $\mathbb{C}^2$ defined by the polynomial equation $F(x,y) = 0$. Set $A(C) := \mathbb{C}[x,y]/(F)$ to be its coordinate ring (Problem 1 in Homework assignment 1). Given a point $p = (a,b) \in C$, let $\mathfrak{m}_p \subset A(C)$ be the maximal ideal $(x-a, y-b)/(F)$ of the point p. Denote the localization of $A(C)$ at $\mathfrak{m}_p$ by $\mathcal{O}_p(C)$. Prove that if p is a smooth point of C, then $\mathcal{O}_p(C)$ is a *discrete valuation ring* (DVR) (see part (ii) of the hint below).

Hint: Use the following basic result from commutative algebra: Let R be an integral domain which is not a field. The following are equivalent:

(i) R is Noetherian and local and the maximal ideal is principal.

(ii) R is a DVR; i.e., there is an irreducible element $t \in R$ such that every non-zero $z \in R$ may be written in the form $z = ut^n$, for a unique pair (u,n), where u is a unit in R and n is a non-negative integer.

Definition: The integer n, in part (ii) of the hint, is called the order of z and we set $ord(z) := n$. Given $G \in \mathbb{C}[x,y]$ we set $ord_p(G_{|C}) := ord(g)$, where g is the image of G in $\mathcal{O}_p(C)$.

(c) Prove that $ord_p(G_{|C})$, in the definition above, is equal to the order of vanishing of the holomorphic function $G_{|C}$ at p. *Hint:* Show first that the latter order is equal to $d \cdot ord_p(G_{|C})$ for some positive integer d, which is independent of G.

(d) If $p \in C$ is a smooth point, where C is $F(x,y) = 0$, then

$$I_p(F,G) \quad = \quad ord_p(G_{|C}). \tag{5}$$

Hint: Prove that the successive quotients $(t^n)/(t^{n+1})$, of the ideals in $\mathcal{O}_p(C)$ generated by powers of the irreducible element t, are all one dimensional. Then use the isomorphism

$$\mathcal{O}_p(\mathbb{C}^2)/(F,G) \to \mathcal{O}_p(C)/(g),$$

which expresses the fact that localization commutes with taking quotients.