Riemann Surfaces Homework Assignment 2 Fall 2004

1. Kirwan section 3.3 page 78: Do problems 3.1, 3.2, 3.3, 3.6.

2. (a) Prove that an irreducible cubic curve in $\mathbb{P}^2$ has at most one singular point (see problem 3.2).

   (b) Do Problem 3.9 from Kirwan's section 3.3 page 78.

   (c) Find the singular points of the cubic curves listed in Problem 3.9. (the third case in the list should be $y^2z - x(x-z)(x-\lambda z)$, for some $\lambda \notin \{0,1\}$. Conclude that an irreducible cubic curve in $\mathbb{P}^2$ is either smooth, has one ordinary double point (a *node*), or one *cusp* (the case $y^2z = x^3$).

3. Kirwan section 3.3 page 78 problem 3.15. This problem will be due only after we cover the material from section 3.2. (Consult also Problem 2.10 from Homework Assignment 1 and the hint given for its solution).

4. Let $U_0 \subset \mathbb{P}^n$ be the Zariski open subset, where $x_0 \neq 0$, and $\varphi_0 : U_0 \to \mathbb{C}^n$ the map given by
$$(a_0, a_1, \ldots, a_n) \quad \mapsto \quad (a_1/a_0, \ldots, a_n/a_0).$$
We have proven in class that φ_0 is a homeomorphism with respect to the classical topologies of $\mathbb{P}^n$ and $\mathbb{C}^n$. Prove that φ_0 is a homeomorphism with respect to the Zariski topologies as well (where the closed subsets are the algebraic subsets).
Hint: Use the maps α and β below to establish a bijection between the closed subsets of U_0 (with respect to the topology induced by the Zariski topology of $\mathbb{P}^n$) and of $\mathbb{C}^n$. Let $S := \mathbb{C}[x_0, x_1, \ldots, x_n]$, $S_d \subset S$ the subset of homogeneous polynomials of degree d, and
$$S^h := \cup_{d \geq 0} S_d$$
the subset of all homogeneous polynomials. Let $\alpha : S^h \to \mathbb{C}[y_1, \ldots, y_n]$ be the map given by
$$f(x_0, \ldots, x_n) \quad \mapsto \quad f(1, y_1, \ldots, y_n).$$
Let $\beta : \mathbb{C}[y_1, \ldots, y_n] \to S^h$ be the map defined by
$$g(y_1, \ldots, y_n) \quad \mapsto \quad x_0^{\deg(g)} g(x_1/x_0, \ldots, x_n/x_0). \tag{1}$$

5. (Based on Hartshorne, Exercise I.2.9) *Projective closure of an affine variety.* Identify $\mathbb{C}^n$ with the Zariski open subset $U_0 \subset \mathbb{P}^n$ via the homeomorphism φ_0 in Problem 4. Given an affine variety $Y \subset \mathbb{C}^n$, we can speak of $\overline{Y}$, the closure of Y in $\mathbb{P}^n$ with respect to the Zariski topology of the latter. $\overline{Y}$ is called the *projective closure of* Y.

   (a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, where β is the map given in equation (1).

   (b) Let $Y \subset \mathbb{C}^n$ be a hypersurface and $f(y_1, \ldots, y_n)$ a generator for $I(Y)$. Show that $\beta(f)$ generates $I(\overline{Y})$.

   (c) Let $Y \subset \mathbb{C}^3$ be the twisted cubic (Problem 2 in Homework Assignment 1). You have shown, that $I(Y)$ is the ideal $(y - x^2, z - x^3)$ in $\mathbb{C}[x, y, z]$. Show that the degree 2 summand of the homogeneous ideal $I(\overline{Y}) \subset \mathbb{C}[w, x, y, z]$ is

three dimensional and find three quadrics spanning it. Use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

(d) Use the three quardics from part 5c to show, that the projective closure of the twisted cubic curve is an embedding of $\mathbb{P}^1$ as a *smooth* curve in $\mathbb{P}^3$.
Note: This is a special case of the *d-Uple embedding* of P^n in P^N, where $N := \binom{n+d}{d} - 1$ (Hartshorne, Exercise I.2.12).

6. (Hartshorne, Exercise I.2.15) *The Quadric Surface in* $\mathbb{P}^3$. Consider the surface Q (i.e., a variety of complex dimension 2) given by the equation $xy - zw = 0$.

   (a) Show the Q is equal to the Segre Embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates (see problem 7 below).

   (b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}$, $\{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$, then L_t and L_u are disjoint, the analogous property for the M_t's, and for all t, u, L_t meets M_u precisely at one point.

   (c) Show that Q contains other curves besides these lines. Deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbb{P}^1$ (where each $\mathbb{P}^1$ has its Zariski topology).

7. (Hartshorne, Exercise I.2.14) *The Segre Embedding.* This problem is **optional**, but make sure to know the statement of this important construction. Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$, in lexicographic order, where $N := rs + r + s$. Note that ψ is well defined and injective. It is called the *Segre Embedding.* Show that the image of ψ is a *subvariety* of $\mathbb{P}^N$.
Hint: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} \ : \ i = 0, \ldots, r, \ \ j = 0, \ldots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $\mathbb{C}[\{z_{ij}\}] \to \mathbb{C}[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which sends z_{ij} to $x_i y_j$. Then show that the image of ψ is $V(\mathfrak{a})$.