Riemann Surfaces Homework Assignment 1 Fall 2004

1. Let $Y \subset \mathbb{C}^n$ be an affine algebraic set. The *affine coordinate ring* of Y is the quotient ring $\mathbb{C}[x_1, x_2, \ldots, x_n]/I(Y)$. It is denoted by $A(Y)$. $A(Y)$ is naturally isomorphic to the ring of functions $\bar{f} : Y \to \mathbb{C}$ on Y, which are the restriction of some polynomial function $f(x_1, \ldots, x_n)$ on $\mathbb{C}^n$. If Y is an affine variety, then $I(Y)$ is a prime ideal and consequently $A(Y)$ is an integral domain (it does not have zero divisors). Points of Y are in one-to-one correspondence with maximal ideals of $A(Y)$ (why?). Hence, if two varieties Y_1 and Y_2 have isomorphic coordinate rings, there is a natural bijection between them. In algebraic geometry, the definition of morphisms (algebraic maps) between affine algebraic varieties is such, that Y_1 and Y_2 are isomorphic, if and only if their coordinate rings are isomorphic as algebras over $\mathbb{C}$.

   (a) Let Y be the plane curve $y = x^2$, i.e., $Y := V(y - x^2)$. Show that $A(Y)$ is isomorphic to the polynomial ring $\mathbb{C}[t]$ in one variable t.

   (b) Let Z be the plane curve $xy = 1$. Show, that $A(Z)$ is not isomorphic to a polynomial ring in one variable.

   (c) Let f be any irreducible quadratic polynomial in $\mathbb{C}[x, y]$, and set $W := V(f)$. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?
   Hint: Write $f(x, y) = q(x, y) + \ell(x, y) + c$, where c is a constant and q, ℓ are homogeneous of degrees 2, 1 respectively. Show that the answer depends only on $q(x, y)$. When we study projective plane curves, we will see that the answer depends on the number of distinct points of intersection of the closure of W with the line at infinity.

2. The *twisted cubic curve.* Let $Y \subset \mathbb{C}^3$ be the set (parametrized curve)

$$\{(t, t^2, t^3) \ : \ t \in \mathbb{C}\}.$$

   Show that Y is a smooth affine variety of dimension 1 as follows:

   (a) Find generators for the ideal $I(Y)$ and show that Y is an algebraic set.

   (b) Show that $A(Y)$ is isomorphic to the polynomial ring $\mathbb{C}[t]$ in one variable. Conclude that Y is an affine variety.

   (c) Show that the tangent space of Y is one dimensional, at every point of Y.

3. Prove that $y^2 - x(x - 1)(x + 1) = 0$ is a smooth and irreducible affine curve.
   Hint: Use the following criterion from your basic modern algebra class for the irreducibility of the cubic polynomial (see, for example, Serge Lang's "Algebra" Chapter 4 Theorem 3.1):

   Eisenstein's Criterion: Let A be a unique factorization domain, $f(x) = a_n x^n + \ \cdots \ a_0$ a polynomial of degree $n \geq 1$ in $A[x]$, and $p \in A$ an irreducible element. Assume further, that

   i) p does not divide a_n,

   ii) p divides a_i, for all $i < n$,

   iii) p^2 does not divide a_0.

   See back ...

Then $f(x)$ is irreducible in the ring $K[x]$, where K is the fraction field of A.

Note: A consequence of Bezout's Theorem (to be proven later) is that any smooth projective plane curve is irreducible. We will (easily) see later, that the closure of $y^2 - x(x-1)(x+1) = 0$ in the projective plane is smooth, providing a geometric proof of the problem.

4. Fix two integers $n \geq 3$ and $d \geq 1$. Show that the Fermat hypersurface in $\mathbb{C}^n$ defined by

$$x_1^d + x_2^d + \cdots + x_n^d = 1$$

is a smooth affine variety.

5. Let Y be the algebraic set in $\mathbb{C}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

6. Do the following problems from Kirwan's book section 2.5 page 46: 2.2 (a), (b), 2.8 (a), (b), 2.9, 2.10 (2.8, 2.9, 2.10 involve curves in projective plane, so they will be due only after the material is covered in class).