Math 621 Homework Assignment 9 Spring 2013

Due: Friday, May 3

1. Lang page 307 Problem 7: The holomorphic automorphism group of a simply connected open set U acts transitively on points. More precisely, let $U \subset \mathbb{C}$ be a simply connected open set, z_1, z_2 two points in U. Use the Riemann-Mapping-Theorem (Lang, page 306) to prove that there exists a holomorphic automorphism f of U such that $f(z_1) = z_2$. *Distinguish the cases when* $U = \mathbb{C}$ *and* $U \neq \mathbb{C}$.

2. Let $\{w_1, w_2\}$ be a set of two complex numbers, linearly independent over $\mathbb{R}$, and $\Lambda := \text{span}_{\mathbb{Z}}\{w_1, w_2\}$ the corresponding lattice. Given a complex number z, denote by $[z]$ its congruence class modulo Λ. Denote by $\mathcal{P}$ the Λ-periodic Weierstrass $\mathcal{P}$-function as well as the induced function

$$\mathcal{P} : \mathbb{C}/\Lambda \to \mathbb{CP}^1.$$

The same abuse of notation will be used for the derivative $\mathcal{P}'$.

(a) Prove that the function $\mathcal{P} : \mathbb{C}/\Lambda \to \mathbb{CP}^1$ is surjective.

(b) Prove that the derivative $\mathcal{P}'$ has precisely three simple zeroes at $[w_1/2]$, $[w_2/2]$, and $[(w_1 + w_2)/2]$.

(c) Let g_2 and g_3 be the constants associated to the lattice Λ in Lang, Ch. XIV, Theorem 2.3 page 399. Let $E \subset \mathbb{C}^2$ be the set of pairs (x, y) of complex numbers satisfying the cubic equation

$$y^2 = 4x^3 - g_2 x - g_3. \tag{1}$$

The image of the function

$$(\mathcal{P}, \mathcal{P}') \ : \ (\mathbb{C}/\Lambda) \setminus \{[0]\} \quad \longrightarrow \quad \mathbb{C}^2$$

is contained in E, by the above mentioned theorem (proven in class). Prove that the map

$$(\mathcal{P}, \mathcal{P}') \ : \ (\mathbb{C}/\Lambda) \setminus \{[0]\} \quad \longrightarrow \quad E$$

is one-to-one and onto.

3. **The elliptic modular function λ.** Let $\mathbb{H} := \{z \ : \ \text{Im}(z) > 0\}$ be the upper half plane. Given a point $\tau \in \mathbb{H}$, we get the lattice $\Lambda_{1,\tau}$ spanned by 1 and τ, and the Weierstarass $\mathcal{P}$ function

$$\mathcal{P} : \mathbb{C}/\Lambda_{1,\tau} \to \mathbb{CP}^1.$$

We have seen that all but four fibers of $\mathcal{P}$ consist of two distinct points, while each point among the origin and the three points of order 2 is the unique point in its fiber. These four points $\{[0], [1/2].[\tau/2], [(1 + \tau)/2]\} \subset \mathbb{C}/\Lambda_{1,\tau}$ are called *ramification points* of $\mathcal{P}$ and the values

$$\{\infty, e_1, e_2, e_3\} \ := \ \{\mathcal{P}(0), \mathcal{P}(1/2), \mathcal{P}(\tau/2), \mathcal{P}((1 + \tau)/2)\} \ \subset \ \mathbb{CP}^1$$

are called *branch points* of $\mathcal{P}$. We get the function

$$\lambda : \mathbb{H} \to \mathbb{C} \setminus \{0, 1\}, \tag{2}$$

sending τ to the cross ratio $(\infty, e_1, e_2, e_3) := \frac{e_3 - e_2}{e_1 - e_2}$ of the four branch points.

Recall that linear fractional transformations, associated to matrices in $SL(2,\mathbb{R})$, act on $\mathbb{H}$ via holomorphic bijections (Homework 6 problem 3). Let $\Gamma \subset SL(2,\mathbb{R})$ be the subgroup of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries, satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2,$$

so that the integers a and d are odd, b and c are even, and $ad - bc = 1$. Your goal is to prove that the function λ, given in Equation (2), is invariant with respect to Γ. Invariance means that

$$\lambda\left(\frac{a\tau + b}{c\tau + d}\right) = \lambda(\tau), \tag{3}$$

for all $\tau \in \mathbb{H}$ and all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

(a) Consider more generally the lattice Λ with generators w_1 and w_2. Set $e_1 := \mathcal{P}(w_1/2)$, $e_2 := \mathcal{P}(w_2/2)$, and $e_3 := \mathcal{P}((w_1 + w_2)/2)$. Prove that the set $\{e_1, e_2, e_3\}$ consist of three distinct values. (Proven in class, but do it again).

(b) Set $\tau := w_2/w_1$. Then $\mathrm{Im}(\tau) \neq 0$. Assume that τ belongs to $\mathbb{H}$. Prove the equality

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}, \tag{4}$$

where λ is given in Equation (2). Caution: The set of three branch points $\{e_1, e_2, e_3\}$ depends on the pair (w_1, w_2), so the set in part 3a may be different from the set in Equation (2). Hint: Use the definition of the Weierstrass $\mathcal{P}$-function in order to show that $e_i(tw_1, tw_2) = t^{-2}e_i(w_1, w_2)$, for all $t \in \mathbb{C}^*$. Conclude that the right hand side of (4) depends only on the ratio w_2/w_1.

(c) Show that the lattice spanned by $w_2' := aw_2 + bw_1$ and $w_1' = cw_2 + dw_1$ is equal to the lattice Λ spanned by w_1 and w_2, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Furthermore, we have the following equalities of congruence classes modulo Λ:

$$[w_1'/2] = [w_1/2] \quad \text{and} \quad [w_2'/2] = [w_2/2].$$

(d) Prove the Γ-invariance of λ stated in Equation (3).

Remark 1: The function λ thus factors through a map $\mathbb{H}/\Gamma \to \mathbb{C} \setminus \{0, 1\}$. It is not hard to show that the latter map is bijective.

Remark 2: Consider the rational function $j(\lambda) := 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$ of problem 11 in homework 3. The composite function $j \circ \lambda : \mathbb{H} \to \mathbb{C}$ is $SL(2,\mathbb{Z})$-invariant.

4. Let $\Lambda \subset \mathbb{C}$ be a lattice generated by two complex numbers w_1 and w_2. Set $t := \lambda(w_2/w_1)$, where λ is given in Equation (2). Prove that there is an affine change of coordinates $(\tilde{x}, \tilde{y}) = (c_1 x + c_0, by)$ mapping the plane cubic given in equation (1) to the cubic

$$\tilde{y}^2 = \tilde{x}(\tilde{x} - 1)(\tilde{x} - t).$$

Hint: Show first that the equality of cross ratios $(\infty, z_1, z_2, z_3) = (\infty, z_1', z_2', z_3')$ implies that the two triples of points are related by an affine linear transformation.