Math 621 Homework Assignment 7 Spring 2013

Due: Friday, April 12

1. (a) Show that single valued analytic branches of $f(z) = z^\alpha$, $\alpha \in \mathbb{C}$, and $f(z) = z^z$ can be defined in any simply connected region, which does not contain the origin.

   (b) (Ahlfors, problem 5 page 148, modified) Let $U \subset \mathbb{C}$ be a connected open subset, such that the points 1 and -1 are in the same connected component of the complement.

   i. Show that a single valued analytic branch of $\sqrt{1-z^2}$ can be defined in U. Hint: Consider first a branch of $\sqrt{\frac{z+1}{z-1}}$.

   ii. Let $\gamma : [0,1] \to U$ be a piecewise smooth path. Choose $\alpha \in \mathbb{C}$, such that $\sin(\alpha) = \gamma(0)$. This is always possible, since $\sin : \mathbb{C} \to \mathbb{C}$ is surjective. Use the local inverse mapping theorem to prove the equality

   $$\sin\left(\alpha + \int_\gamma \frac{dz}{\sqrt{1-z^2}}\right) = \gamma(1)$$

   for one of the two branches of $\sqrt{1-z^2}$. Conclude that when γ is closed, the integral $\int_\gamma \frac{dz}{\sqrt{1-z^2}}$, for either branch of $\sqrt{1-z^2}$, is the difference between two values of $\arcsin(\gamma(0))$.

   iii. Let γ be a closed curve in U. Prove that $\int_\gamma \frac{dz}{\sqrt{1-z^2}} = 2k\pi$, for some integer k. Caution: If $\sin(w_1) = \sin(w_2)$, then $w_2 = w_1 + 2k\pi$, **OR** $w_2 = -w_1 + (2k+1)\pi$, $k \in \mathbb{Z}$. **Hint:** Set $Z := \mathbb{C} \setminus \{\frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z}\}$, $Y := \mathbb{C} \setminus \{0, i, -i\}$, and $X := \mathbb{C} \setminus \{1, -1\}$. The function $\sin : Z \to X$ factors as the composition $\sin(w) = q(e^{iw})$, where the function $q : Y \to X$ is given by $q(\zeta) = \frac{\zeta - \frac{1}{\zeta}}{2i}$. Let $g(z) := \sqrt{1-z^2}$ be one of the two branches constructed in part 1(b)i. Set $h(z) = g(z) + iz$. Then $h : U \to Y$ is a section of q, i.e., $q(h(z)) = z$. Prove first the equality

   $$\int_\gamma \frac{dz}{\sqrt{1-z^2}} = \pm i \int_{h(\gamma)} \frac{d\zeta}{\zeta}.$$

2. Laurent Serries: Lang page 164: 8, 12, 13

3. Consider the Laurent series
   $\tan(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, which is valid in the annulus $\frac{\pi}{2} < |z| < \frac{3\pi}{2}$. Find the coefficients a_n with index $-\infty < n \leq -1$. Hint: Use integration.

4. Isolated Singularities: Lang page 170: 1a,c,e, 4

5. Find the value of the integral $\int_C \frac{3z^3+2}{(z-1)(z^2+9)} dz$ for:

   (a) C the circle $|z-2| = 2$.

   (b) C the circle $|z| = 4$.

6. Suppose that $f(z) = \frac{g(z)}{h(z)}$, $g(z_0) \neq 0$ and $h(z)$ has a zero of order 2 at z_0. Prove that

$$Res_{z_0} f \;=\; \frac{2g'(z_0)}{h''(z_0)} - \frac{2g(z_0)h'''(z_0)}{3(h''(z_0))^2}$$

7. Compute the integral of the following functions over the circle $|z| = 2$:

   (a) $f(z) = \dfrac{1}{(z-3)(1+2z)^2(1-3z)^3}$

   (b) $f(z) = \dfrac{z^2}{1-e^{z/4}}$

   (c) $f(z) = \dfrac{\cos(1/z)}{1+z^4}$

8. Let $P(z)$ and $Q(z)$ be polynomials and suppose that $\deg Q \geq \deg P + 2$. Show that

$$\sum_{\zeta} Res_{\zeta}(P/Q) = 0,$$

   where the sum runs over all singularities of the rational function P/Q. Do this problem in two ways: (1) Directly, without considering the residue at ∞. (2) As a special case of problem 10.

9. Lang Problem 37 page 190: Let f be analytic on $\mathbb{C}$ with exception of a finite number of isolated singularities $\{z_1, \ldots z_n\}$. Define the *residue at infinity*

$$Res_{\infty} f(z)dz \;:=\; -\frac{1}{2\pi i}\int_{|z|=R} f(z)dz$$

   for $R > \max\{|z_1|, \ldots |z_n|\}$.

   (a) Show that $Res_{\infty} f(z)dz$ is independent of R.

   (b) Show that the sum of the residues of f in the extended complex plane $\mathbb{CP}^1$ is equal to zero. (This result is often refered to as *The residue Theorem.*)

10. Lang Problem 38 page 190 (Cauchy's Residue Formula on the Riemann Sphere).

11. Basic Exam, September 1998 Problem 5: Compute $\displaystyle\int_C \frac{z^4 e^{1/z}}{1-z^4}dz$ where C denotes the circle $\{|z| = 2\}$ transversed counterclockwise. *Hint: Use the residue at infinity (problem 10) to save computations.*

12. Basic Exam, January 2000 Problem 7: Let f be a one-to-one holomorphic map from a region D_1 onto a region D_2. Suppose that D_1 contains the closure of the disk $\Delta := \{|z - z_0| < \rho\}$. Prove that for $w \in f(\Delta)$ the inverse function $f^{-1}(w)$ is given by

$$f^{-1}(w) \;=\; \frac{1}{2\pi i}\int_{\{|z-z_0|=\rho\}} \frac{f'(z)}{f(z)-w} \cdot z dz$$

13. Let $P(z)$ be a polynomial of degree d and assume that the roots $\zeta_1, \ldots, \zeta_d$ of P are simple. Show that for R sufficiently large:

$$\int_{\{|z|=R\}} \frac{z^k P'(z)}{P(z)}dz \;=\; \sum_{i=1}^{d} \zeta_i^k.$$