Math 621 Homework Assignment 6 Spring 2013

Due: Friday, March 29

1. Find the maximum value of the function $g(z) = |z^3 - z|$ on the disk $|z| \leq 2$. Justify your answer!

2. Lang page 213 Problem 1: Let f be analytic on the unit disc D, and assume that $|f(z)| < 1$ on the disc. Prove that if there exist two distinct points a, b in the disc, which are fixed under f (that is $f(a) = a$ and $f(b) = b$), then $f(z) = z$.

3. Lang, page 219 problem 8: Use Schwarz's Lemma to prove that $PSL(2, \mathbb{R})$ is the group $\text{Aut}(\mathbb{H})$ of holomorphic automorphisms of the upper half plane. ($PSL(2, \mathbb{R})$ is naturally identified with the group of fractional linear transformations which are associated to invertible 2×2 matrices with *real* coefficients and determinant 1). *Hint:* (a) Show that $PSL(2, \mathbb{R})$ is an index 2 subgroup of $PGL(2, \mathbb{R})$. (b) Use Schwarz's Lemma to show that if f belongs to $\text{Aut}(\mathbb{H})$, then f is a linear fractional transformation. (c) Show that if f belongs to $\text{Aut}(\mathbb{H})$, then it belongs to $PGL(2, \mathbb{R})$. (d) Show that if f belongs to $PGL(2, \mathbb{R})$, then f map $\mathbb{H}$ either to $\mathbb{H}$ or to the lower-half-plane. (f) Show that if f belongs to $PGL(2, \mathbb{R})$, then $f(\mathbb{H}) = \mathbb{H}$, if and only if f belongs to $PSL(2, \mathbb{R})$ (calculate $f'(x)$, for $x \in \mathbb{R}$).

4. Lang page 213 Problem 2: Let $f : D \to D$ be a holomorphic map from the disc into itself. Prove that, for all $a \in D$, we have
$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$
Moreover, *equality for some a implies that f is a linear fractional transformation. Hint:* Let g be an automorphism of D such that $g(0) = a$, and let h be the automorphism which maps $f(a)$ on 0. Let $F = h \circ f \circ g$. Compute $F'(0)$ and apply the Schwarz Lemma.

5. Ahlfors, page 136 Problem 2: Let $f(z)$ be analytic and $\text{Im}(f(z)) \geq 0$ for all z in the upper half plane $\mathbb{H}$. Show that for $z, z_0 \in \mathbb{H}$,
$$\left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| \leq \frac{|z - z_0|}{|z - \bar{z}_0|}$$
and, writing $z = x + iy$,
$$\frac{|f'(z)|}{\text{Im} f(z)} \leq \frac{1}{y}.$$
Moreover, equality, in either one of the two inequalities above, implies that f is a linear fractional transformation.

6. Ahlfors, page 136 Problem 6: If γ is a path, piecewise of type C^1, contained in the open unit disc D, then the integral
$$\int_\gamma \frac{|dz|}{1 - |z|^2}$$
is called the *non-euclidean length* or *hyperbolic length* of γ. Let $f : D \to D$ be an analytic function from the disc into itself. Show that f maps every γ on a path with smaller or equal non-euclidean length. Deduce that a linear fractional transformation from D onto itself preserves non-euclidean lengths.

7. Ahlfors, page 136 Problem 7: (Modified) It can be shown, that the path of smallest non-euclidean length, joining the origin 0 to a point $z \in D$, is the straight line segment between them.

   (a) Use this fact to show that the path of smallest non-euclidean length, that joins two given points in the unit disk, is the piece of the circle C which is orthogonal to the unit circle ∂D. The shortest non-euclidean length is called the *non-euclidean distance.*

   (b) Show that the non-euclidean distance between z_1 and z_2 is

$$\frac{1}{2}\log\frac{1+\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|}{1-\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|}$$