Math 621 Homework Assignment 5 Spring 2013

Due: Friday, March 15

1. Lang page 132 Problem 1: Find the integrals over the unit circle C:

   (a) $\displaystyle\int_C \frac{\cos(z)}{z} dz$ (b) $\displaystyle\int_C \frac{\sin(z)}{z} dz$ (c) $\displaystyle\int_C \frac{\cos(z^2)}{z} dz$

2. Ahlfors page 120 Problem 3: Compute $\displaystyle\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$ under the condition $|a| \neq \rho$.
   Hint: Make use of the equations $z\bar{z} = \rho^2$ and $|dz| = -i\rho\frac{dz}{z}$.

3. Show that the successive derivatives of an analytic function at a point can never satisfy $\left|f^{(n)}(z)\right| > n!n^n$ in two ways: (a) Using Cauchy's Estimate. (b) Using Taylor's Theorem.

4. Lang page 132 Problem 3 (modified): Let f be an entire function, k a positive integer, and let $\| f \|_R$ be the maximum of $|f|$ on the circle of radius R centered at the origin. Then f is a polynomial of degree $\leq k$ if and only if there exist constants C and $R_0 \geq 0$ such that

   $$\| f \|_R \leq CR^k,$$

   for all $R \geq R_0$. (Note: one direction was proven in HW 1 Problem 8).

5. Let $\tau \in \mathbb{C}$ be a complex number and assume that $\mathrm{Im}(\tau) \neq 0$. A function f is said to be *doubly periodic with periods* 1 *and* τ if

   $$f(z+1) = f(z) \quad \text{and} \quad f(z+\tau) = f(z), \quad \text{for all } z \in \mathbb{C}.$$

   Show that every entire function, which is doubly periodic with periods 1 and τ, is necessarily constant. (We will see that there exist non-constant, doubly periodic, meromorphic functions $f : \mathbb{C} \to \mathbb{CP}^1$).

6. Lang page 159 Problem 7: Let f be analytic on a closed disc $\overline{D}$ of radius $b > 0$, centered at z_0. Show that

   $$\frac{1}{\pi b^2} \int\int_D f(x+iy) dy dx = f(z_0).$$

   Hint: Use polar coordinates and Cauchy's Formula.

7. Lang page 159 Problem 9 (modified): Let f be analytic and $1:1$ on the unit disk D, and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor series expansion of f. Show that

   $$\mathrm{area} f(D) = \pi \sum_{n=0}^{\infty} n|a_n|^2.$$

8. (a) Ahlfors, page 130 Problem 2: Show that a function which is analytic in the whole plane and has a non-essential singularity at ∞ reduces to a polynomial. (You may use Problem 4 above).

(b) Lang, page 171 Problem 10: Show that any function, which is meromorphic on the extended complex plane, is a rational function.

9. (a) Show that the functions $\cos(z)$ and $\sin(z)$ have essential singularities at ∞.

   (b) Let $f(z) = \cos\left(\frac{1+z}{1-z}\right)$, $|z| < 1$. Find the set Z_f of zeroes of f. Does Z_f have any accumulation points? Explain. (See Lang, page 21 for the definition of an *accumulation point*).

10. Lang, page 171 Problem 11: Define the order $\mathrm{Ord}_p f$ of a meromorphic function f at a point p to be $\mathrm{Ord}_p f := \begin{cases} m & \text{if } p \text{ is a zero of } f \text{ of order } m \\ -m & \text{if } p \text{ is a pole of } f \text{ of order } m \end{cases}$

    Above, m could be zero, meaning that f is analytic at p and $f(p) \neq 0$.

    Let f be a meromorphic function on the extended complex plane $\mathbb{CP}^1$ (so a rational function by problem 8a).

    (a) Prove that $\sum_{p\in\mathbb{CP}^1} \mathrm{Ord}_p f = 0.$

    In other words, the number of points in the fiber $f^{-1}(0)$, counted with multiplicity, is equal to the number of points in $f^{-1}(\infty)$, counted with multiplicity.

    (b) Prove that all fibers $f^{-1}(\lambda)$, $\lambda \in \mathbb{CP}^1$, of f consist of the same number of points, provided they are counted with multiplicity,

11. Ahlfors, page 130 Problem 5: Let z_0 be an isolated singularity of an analytic function f. Prove that if $\mathrm{Re}(f)$ is bounded from above or below, then z_0 is a removable singularity. *Ahlfors' Hint:* Apply a linear l.f.t. *Note:* Personally, I find it easier to avoid using a l.f.t (which does not seem to help rule-out the case of a pole). Instead, a short proof can be obtained using both the Casorati-Weirstrass and the Open Mapping Theorems.

12. (a) Set $f(z) := e^{(z^2)}$. Prove that there exists a unique entire function g, which is an anti-derivative of f satisfying $g(0) = 0$.

    (b) Show that $g : \mathbb{C} \to \mathbb{C}$ is a *local homeomorphism*, i.e., for every $\lambda \in \mathbb{C}$, there exist open neighborhoods U of λ and V of $g(\lambda)$, such that $g : U \to V$ is a homeomorphism (one to one and onto with a continuous inverse).

    (c) Use Picard's Theorem stated below in order to prove that g is surjective. Hint: Note that g is an odd function.

    (d) Prove that $g : \mathbb{C} \to \mathbb{C}$ is not injective. If you took Math 671, conclude that g is not a covering map.[1]

    Picard's Theorem (Ahlfors Theorem 5 page 307): Let g be a non-constant entire function. Then the image $g(\mathbb{C})$ of g is either the whole of $\mathbb{C}$, or $\mathbb{C} \setminus \{\lambda\}$, for a single complex number λ.

[1] Let $D_\delta(\lambda)$ denote the open disk of radius δ centered at λ. The map g is not a covering means that there exists a point $\lambda \in \mathbb{C}$, such that for every $\delta > 0$ the inverse image $g^{-1}(D_\delta(\lambda))$ has a connected component U, such that $g : U \to D_\delta(\lambda)$ is not a homeomorphism.