Math 621 Homework Assignment 4 Spring 2013

Due: Friday, March 8

1. Lang problem 4 page 102.

2. (a) Describe the curve C parametrized by $\gamma(t) = a\cos(t) + ib\sin(t)$, $t \in [0, 2\pi]$. Compute $\int_C \frac{dz}{z}$.

   (b) Compute $\int_0^{2\pi} \frac{dt}{a^2\cos^2(t) + b^2\sin^2(t)}$.
   Hint: Use $\cos(t) = \frac{e^{it}+e^{-it}}{2}$ and $\sin(t) = \frac{e^{it}-e^{-it}}{2i}$ to convert to an integral of a rational function over the unit circle.

3. (a) Let S_R denote the semi-circle

$$S_R \; := \; \{Re^{i\theta} \; : \; 0 \leq \theta \leq \pi\}.$$

   Show that $\lim_{R\to\infty} \int_{S_R} \frac{e^{iz}}{z} dz = 0$.

   (b) Let $\alpha, \beta \in \mathbb{C}$ be such that $Re(\alpha) \leq 0$ and $Re(\beta) \leq 0$. Show that

$$\left|e^{\alpha} - e^{\beta}\right| \leq |\beta - \alpha|.$$

4. Use Green's Theorem to prove a *weaker* version of Cauchy-Goursat's Theorem for a rectangle:

   Let f be a holomorphic function defined and having a **continuous** *derivative f' in an open set U containing a rectangle R. Then*

$$\int_{\partial R} f dz \; = \; 0.$$

   Recall the statement of Green's Theorem: Let γ be an oriented piecewise smooth *simple* path (i.e., each connected component of γ does not intersect itself) in the plane. Assume that γ bounds a region D (and has the *induced orientation*, i.e., each smooth piece of γ is oriented so that D is on the left as you move along γ). Let $p(x, y)$, $q(x, y)$ be two functions which are defined and have **continuous partial derivatives** in an open set $U \subset \mathbb{R}^2$ containing D and γ. Then

$$\int_\gamma p dx + q dy \; = \; \int\int_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx dy.$$

5. Let U_1 and U_2 be open subsets of $\mathbb{C}$ and $f : U_1 \to U_2$ a holomorphic function. (You may assume the continuity of f'). Let $\gamma : [a, b] \to U_1$ be a path and set $\gamma_2 := f \circ \gamma_1$. Let g be a continuous complex valued function on U_2. Prove the equality

$$\int_{\gamma_2} g(z)dz = \int_{\gamma_1} g(f(z))f'(z)dz.$$

6. Ahlforse page 108 problem 5: Suppose that $f(z)$ is analytic on a closed curve γ (i.e., f is analytic in an open set containing γ). Show that $\int_\gamma \overline{f(z)} f'(z) dz$ is purely imaginary. (You may assume the continuity of f').

7. Ahlforse page 108 problem 6: Assume that $f(z)$ is analytic and satisfies the equality $|f(z) - 1| < 1$ in a region (connected open set) Ω. Show that $\int_\gamma \frac{f'(z)}{f(z)} dz = 0$, for every closed curve γ in Ω. (You may assume the continuity of f').

8. (a) Let D be an open disk in $\mathbb{C}$ and let f be continuous in D. Suppose that $\int_{\partial R} f(z) dz = 0$ for every closed rectangle R contained in D. Prove that f is holomorphic.

   (b) Suppose that f is continuous in all of $\mathbb{C}$ and holomorphic in $\mathbb{C} \setminus \mathbb{R}$. Prove that f is holomorphic everywhere.

9. Let U be an open subset of $\mathbb{C}$ and f_n a sequence of holomorphic functions which converges, uniformly on compact subsets of U, to a function f. Prove that f is holomorphic in U and that f_n' converges, uniformly on compact subsets of U, to f'.