Math 621 Homework Assignment 3 Spring 2013

Due: Friday, February 22

1. Find a fractional linear transformation that maps
   (a) $0, 1, \infty$ to $1, -1, 0$,
   (b) $0, i, -i$ to $1, -1, 0$.

2. Let $T(z) = \frac{z-i}{z+i}$. Determine the image of horizontal lines $Im(z) = b$ under T. When the image is a circle, determine the center and radius (you may find it helpful to use the notion of *symmetry*).

3. (a) Find a function $H(x, y)$ harmonic in the domain $D := \{(x, y) \ : \ 0 < y < 1\}$ and such that $H \equiv 0$ on the line $\{y = 0\}$ and $H \equiv 1$ on the line $y = 1$.
   (b) Find a function $G(x, y)$ harmonic in the region inside the circle $\{|z| = 2\}$ and outside the circle $\{|z + 1| = 1\}$ and such that $G \equiv 0$ on the inner circle and $G \equiv 1$ on the outer circle. *Hint:* See HW 2 problem 1(a).

4. Lang page 238-239 problems 12 part c (interpret the result geometrically), 14 parts b,c, 13 part c (classify the fixed points according to Problem 14).

5. Given a circle or a straight line C, denote by R_C the reflection with respect to C. We have seen that the following equality holds for every linear fractional transformation f:
$$R_{f(C)} \circ f = f \circ R_C. \tag{1}$$
   (a) Let G be the group of bijective maps from $\mathbb{CP}^1$ to itself generated by linear fractional transformations and reflections R_C as above. Given $g \in G$, prove that either g is a fractional linear transformation, or $f(z) := g(\bar{z})$ is a fractional linear transformation. Conclude that the group $PGL(2, \mathbb{C})$ of fractional linear transformations is an index two subgroup of G.
   (b) Prove that an element $g \in G$ is a conformal map, i.e., it preserves angles.
   (c) Prove that equation (1) holds more generally for $f \in G$.
   (d) Let C be a circle or a straight line and f an element of G. Assume that ∞ does not belong to $f(C)$. Use Equation (1) to prove that $f(C)$ is a circle centered at $f(R_C(f^{-1}(\infty)))$.

6. Ahlfors page 83 problem 2: Reflect the imaginary axis, the line $x = y$, and the circle $|z| = 1$ in the circle $|z - 2| = 1$.

7. Find the linear fractional transformation which carries the circle $|z| = 3$ into $|z - 1| = 1$, the point $3i$ to the origin, and the origin to i.

8. Ahlfors page 33 problem 4: What is the general form of a rational function (of arbitrary degree) which has absolute value 1 on the circle $|z| = 1$? In particular, how are the zeros and poles related to each other? (You may use the Fundamental Theorem of Algebra stated in Corollary 7.6 page 130 in Lang. We will prove it later in the course). *Hint:* Prove first that the rational function f satisfies $f = R \circ f \circ R$, where R is the reflection with respect to the unit circle.

9. Ahlfors page 33 problem 5: If a rational function is real on $|z| = 1$, how are the zeros and poles situated?

10. Let S_3 be the permutation group of the set $\{0, 1, \infty\}$. For each permutation σ, denote by T_σ the fractional linear transformation taking $0, 1, \infty$ to $\sigma(0), \sigma(1), \sigma(\infty)$.

    (a) Find the six l.f.t $\{T_\sigma \ : \ \sigma \in S_3\}$.

    (b) The orbit of λ is the set $\{T_\sigma(\lambda) \ : \ \sigma \in S_3\}$. Show that all but two S_3 orbits in $\mathbb{C} \setminus \{0, 1\}$ consist of six elements. One special S_3 orbit in $\mathbb{C} \setminus \{0, 1\}$ consists of two elements and the other special orbit consists of three elements.

11. We have seen that the cross ratios $(z_1' : z_2' : z_3' : z_4')$ and $(z_1'' : z_2'' : z_3'' : z_4'')$ are equal, if and only if there exists a linear fractional transformation mapping the first ordered 4-tuple to the second. We work out the analogous statement for unordered sets of 4 distinct points. Let j be the rational function (of degree 6)

$$j(\lambda) \ = \ 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$$

    The composition $j(z_1 : z_2 : z_3 : z_4)$, of j and the cross ratio, is called the *j-invariant* of the unordered set $\{z_1, z_2, z_3, z_4\}$. Use your answer to problem 10 to show

    (a) The function $j(z_1 : z_2 : z_3 : z_4)$ is symmetric in the z_i. (Find an argument avoiding a tedious check).

    (b) There exists a linear fractional transformation mapping the *unordered* set $\{z_1', z_2', z_3', z_4'\}$ onto $\{z_1'', z_2'', z_3'', z_4''\}$ if and only if their j-invariants are equal.

12. Lang Ch III Sec 2 page 102 problems 5, 7